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On critical logarithmic double phase problems with locally defined perturbation

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ABSTRACT

This paper deals with critical logarithmic double phase problems of the form

$$-\operatorname{div}\mathcal{K}(u)=g(x,u)+|u|^{p^*-2}u\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega,$$

where $\operatorname{div}\mathcal{K}$ is the logarithmic double phase operator defined by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right),$$

e is Euler's number, $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $1<p<N$, $p<q<p^*=\frac{Np}{N-p}$, $0\leq\mu(\cdot)\in L^\infty(\Omega)$ and $g\colon\Omega\times[-\xi,\xi]\to\mathbb{R}$ for $\xi>0$ is a locally defined Carathéodory function satisfying a certain behavior near the origin. Based on appropriate truncation techniques and a suitable auxiliary problem, we prove the existence of a whole sequence of sign-changing solutions of the problem above which converges to 0 in the logarithmic Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and in $L^\infty(\Omega)$.

1. Introduction

In the recent work by Arora–Crespo-Blanco–Winkert [5], the authors introduced and studied the properties of the functional

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$$I(u)=\int\limits_{\Omega}\left(\frac{|\nabla u|^p}{p}+\mu(x)\frac{|\nabla u|^q}{q}\log(e+|\nabla u|)\right)\mathrm{d}x \tag{1.1}$$

and of the corresponding logarithmic double phase operator given by

$$\operatorname{div}\mathcal{K}(u)=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right), \tag{1.2}$$

with $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ being the related logarithmic Musielak-Orlicz Sobolev space while

$$\mathcal{H}_{\log}(x,t)=t^p+\mu(x)t^q\log(e+t)\quad\text{for all }(x,t)\in\overline{\Omega}\times[0,\infty),$$

for $1<p<N$, $p<q$ and $0\le\mu(\cdot)\in L^\infty(\Omega)$. In the past, special cases of the functional given in (1.1) have been investigated. The local Hölder continuity of the gradient of local minimizers of

$$u\mapsto\int\limits_{\Omega}\Big[|\nabla u|^p+\mu(x)|\nabla u|^p\log(e+|\nabla u|)\Big]\,\mathrm{d}x,$$

($p=q$ in (1.1)), was shown by Baroni–Colombo–Mingione [7] for $1<p<\infty$ and $0\le\mu(\cdot)\in C^{0,\alpha}(\overline{\Omega})$ while in a more recent work by De Filippis–Mingione [9], the local Hölder continuity of the gradient of local minimizers of the functional

$$u\mapsto\int\limits_{\Omega}\big[|\nabla u|\log(1+|\nabla u|)+\mu(x)|\nabla u|^q\big]\,\mathrm{d}x, \tag{1.3}$$

has been examined whenever $0\le\mu(\cdot)\in C^{0,\alpha}(\overline{\Omega})$ and $1<q<1+\frac{\alpha}{n}$. It should be noted that (1.3) originates from functionals with nearly linear growth of the form

$$u\mapsto\int\limits_{\Omega}|\nabla u|\log(1+|\nabla u|)\,\mathrm{d}x, \tag{1.4}$$

which has been studied, for example, in the papers by Fuchs–Mingione [13] and Marcellini–Papi [25]. We point out that (1.4) occur in the theory of plasticity with logarithmic hardening, see, for example, Seregin–Frehse [34] and Fuchs–Seregin [14]. Moreover, the famous work of Marcellini [24] includes as a special case functionals with logarithmic term of the form

$$u\mapsto\int\limits_{\Omega}(1+|\nabla u|^2)^{\frac{p}{2}}\log(1+|\nabla u|)\,\mathrm{d}x.$$

In the present work we study critical elliptic problems involving the logarithmic double phase operator given in (1.2). To be more precise, given a bounded domain $\Omega\subseteq\mathbb{R}^N$, $N\ge 2$, with Lipschitz boundary $\partial\Omega$, we investigate the Dirichlet problem

$$-\operatorname{div}\mathcal{K}(u)=g(x,u)+|u|^{p^*-2}u\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega, \tag{1.5}$$

where $\operatorname{div}\mathcal{K}$ is as in (1.2) while we suppose the following assumptions on the exponents p,q, the weight function $\mu(\cdot)$ and the perturbation $g(\cdot,\cdot)$:

(A1) $1<p<N$, $p<q<p^*:=\frac{Np}{N-p}$ and $0\le\mu(\cdot)\in L^\infty(\Omega)$;

(A2) $g\colon \Omega \times [-\xi, \xi] \to \mathbb{R}$ is a Carathéodory function for fixed $\xi > 0$ with $g(x,0) = 0$ and $g(x,\cdot)$ is odd for a.a. $x \in \Omega$;

(A3) there exists $\eta \in L^\infty(\Omega)$ such that

$$|g(x,s)| \leq \eta(x) \quad \text{for a.a.}\, x \in \Omega \text{ and for all } |s| \leq \xi;$$

(A4) there exists $\gamma \in \left(1, \min\left\{p, \frac{p^2}{N-p} + 1\right\}\right)$ such that

$$\lim_{s \to 0} \frac{g(x,s)}{|s|^{\gamma-2}s} = 0 \quad \text{uniformly for a.a.}\, x \in \Omega;$$

(A5)

$$\lim_{s \to 0} \frac{g(x,s)}{|s|^{p-2}s} = +\infty \quad \text{uniformly for a.a.}\, x \in \Omega.$$

We call a function $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ a weak solution of problem (1.5) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)\left(\log(e + |\nabla u|) + \frac{|\nabla u|}{q(e + |\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right) \cdot \nabla\varphi \,\mathrm{d}x$$
$$= \int_\Omega \left(g(x,u) + |u|^{p^*-2}u\right)\varphi \,\mathrm{d}x$$

is satisfied for all $\varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

Our main result reads as follows.

Theorem 1.1. *Suppose the assumptions* (A1)–(A5), *then problem* (1.5) *admits a sequence* $\{w_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *of sign-changing solutions such that* $\|w_n\| \to 0$ *and* $\|w_n\|_\infty \to 0$ *as* $n \to \infty$, *where* $\|\cdot\|$ *and* $\|\cdot\|_\infty$ *are the norms in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ *and in* $L^\infty(\Omega)$, *respectively.*

We point out that the right-hand side of (1.5) consists of the combined effect of a locally defined Carathéodory perturbation $g(x,\cdot)$ along with the critical term $u \to |u|^{p^*-2}u$ with $p^* := \frac{Np}{N-p}$ being the critical exponent related to the given number $1 < p < N$. The main difficulty in the study of (1.5) is the appearance of the critical term and the lack of compactness. In order to overcome this fact, we are going to study an appropriate auxiliary problem by using suitable truncation functions which makes the auxiliary problem coercive. Then we are able to show the existence of extremal constant sign solutions of this auxiliary problem which will be used in order to apply the symmetric mountain pass theorem due to Kajikiya [19]. With our work, we are not only extending the work of Liu–Papageorgiou [22] from the double phase setting to the logarithmic double phase one, but we are also in the position to weaken the assumptions so that assumption H_1 (iii) in [22] is not needed anymore. For additional information and details we also refer to Papageorgiou–Vetro–Winkert [29] in which the double phase problem with variable exponent has been discussed.

As mentioned at the beginning of the Introduction, the logarithmic double phase operator (1.2) has been recently introduced and so only few papers exist involving such operator. The first one has been published by Arora–Crespo-Blanco–Winkert [5] who treated the problem

$$-\operatorname{div}\mathcal{K}(u) = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\operatorname{div}\mathcal{K}$ is as in (1.2) but with variable exponents and with a Carathéodory function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ having subcritical growth and a certain behavior at infinity and near the origin. The authors prove the existence of a least energy sign-changing solution by minimization of the related energy function over the corresponding Nehari manifold of (1.6) under the stronger assumption that $q+1<p^*$. We also refer to a recent work by the same authors [4] concerning new embeddings and existence results. Another logarithmic double phase operator different from the one in (1.2) has been introduced by Vetro–Zeng [40] who studied existence and uniqueness of equations involving the operator

$$u\mapsto \Delta_{\mathcal{H}_L}u=\operatorname{div}\left(\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right),\quad u\in W_0^{1,\mathcal{H}_L}(\Omega),$$

where $\mathcal{H}_L\colon \Omega\times[0,\infty)\to[0,\infty)$ is given by

$$\mathcal{H}_L(x,t)=(t^p+\mu(x)t^q)\log(e+t),$$

with $1<p<q$ and $\mathcal{H}_L'$ stands for the derivative of $\mathcal{H}_L$ with respect to the second variable. The operator (1.2) also appeared in the work by Vetro–Winkert [39] who proved the boundedness, closedness and compactness of the solution set to the problem

$$-\operatorname{div}\mathcal{K}(u)=f(x,u,\nabla u)\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega,$$

where $\operatorname{div}\mathcal{K}$ is as in (1.2) but with variable exponents and $f\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a convection term with very mild structures conditions. Finally, the operator in [39] is also involved in a Kirchhoff type context by Vetro [38].

We also mention the recent work by Tran–Nguyen [37] who showed existence results for equations involving (1.2) when $p=q$. In addition, we also refer to some works dealing with logarithmic perturbations on the right-hand side for Schrödinger equations or p-Laplace problems. In 2009, Montenegro–de Queiroz [26] studied the problem

$$-\Delta u=\chi_{u>0}(\log(u)+\lambda f(x,u))\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega,\tag{1.7}$$

with a function $f(x,\cdot)$ being nondecreasing and sublinear while f_u is supposed to be continuous. They showed that problem (1.7) has a maximal solution $u_\lambda\geq 0$ of type $C^{1,\gamma}(\overline{\Omega})$. We also refer to the works by Figueiredo–Montenegro–Stapenhorst [11,12] who considered a similar problem in planar domains with f being of exponential growth. Furthermore, logarithmic Schrödinger equations of the form

$$-\Delta u+V(x)u=Q(x)u\log(u^2)\quad\text{in }\mathbb{R}^N\tag{1.8}$$

have been studied by Squassina–Szulkin [36] who showed the existence of infinitely many solutions of (1.8). More results for logarithmic Schrödinger equations have been published by Alves–de Morais Filho [2], Alves–Ji [3] and Shuai [35], see also Alves–da Silva [1] about logarithmic Schrödinger equations on exterior domains and Bahrouni–Fiscella–Winkert [6] for sign-changing potentials in $\mathbb{R}^N$. Finally, we mention some related works for double phase problems without logarithmic terms, see the papers by Ge–Pucci [15], Guo–Liang–Lin–Pucci [16], and Liu–Pucci [23].

The paper is organized as follows. In Section 2 we recall the main properties of the logarithmic Musielak-Orlicz Sobolev spaces and the logarithmic double phase operator (1.2). Moreover, we point out the main results about the eigenvalue problem of the p-Laplacian with homogeneous Dirichlet boundary condition. In Section 3 we first study an auxiliary problem and prove the existence of extremal constant sign solutions and then we apply the results of Kajikiya [19] to give the proof of Theorem 1.1.

2. Preliminaries

This section is devoted to the main properties of logarithmic Musielak-Orlicz Sobolev spaces, the corresponding logarithmic double phase operator and some tools which will be used in the sequel. Most of the results are taken from the recent paper by Arora–Crespo-Blanco–Winkert [5]. We also refer to the monographs by Diening–Harjulehto–Hästö–Růžička [10], Harjulehto–Hästö [17], Papageorgiou–Winkert [30] and the paper by Crespo-Blanco–Gasiński–Harjulehto–Winkert [8]. First, for $1 \leq r \leq \infty$, $L^r(\Omega)$ stands for the usual Lebesgue space with norm $\|\cdot\|_r$ while $W_0^{1,r}(\Omega)$ denotes the related Sobolev space with zero traces endowed with the equivalent norm $\|\nabla \cdot\|_r$ for $1 < r < \infty$.

Now, we introduce the nonlinear map $\mathcal{H}_{\log}\colon \overline{\Omega} \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^p + \mu(x) t^q \log(e+t),$$

where we assume hypothesis (A1). Denoting by $M(\Omega)$ the set of all measurable function $u\colon \Omega \to \mathbb{R}$, we can introduce the space $L^{\mathcal{H}_{\log}}(\Omega)$ by

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{ u \in M(\Omega)\colon \rho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x < \infty \right\},$$

equipped with the norm

$$\|u\|_{\mathcal{H}_{\log}} := \inf\left\{\lambda > 0\colon \rho_{\mathcal{H}_{\log}}\left(\frac{u}{\lambda}\right) \leq 1\right\} \quad \text{for } u \in L^{\mathcal{H}_{\log}}(\Omega),$$

where $\rho_{\mathcal{H}_{\log}}$ is called modular function corresponding to $\mathcal{H}_{\log}$. We know that $L^{\mathcal{H}_{\log}}(\Omega)$ is a separable and reflexive Banach space. The corresponding logarithmic Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ is then given by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{u \in L^{\mathcal{H}_{\log}}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}}.$$

Furthermore, we set

$$W_0^{1,\mathcal{H}_{\log}}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_{\log}}}.$$

Note that both spaces $W^{1,\mathcal{H}_{\log}}(\Omega)$ and $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ are separable, reflexive Banach spaces. In addition, we can equip the space $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ with the equivalent norm

$$\|u\| := \|\nabla u\|_{\mathcal{H}_{\log}},$$

see Arora–Crespo-Blanco–Winkert [5, Proposition 3.9]. In the following, we use the abbreviations $\rho_{\mathcal{H}_{\log}}(\nabla u) := \rho_{\mathcal{H}_{\log}}(|\nabla u|)$ and $\mathcal{H}_{\log}(\,\cdot\,,\nabla u) = \mathcal{H}_{\log}(\,\cdot\,,|\nabla u|)$ for $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

The following embedding results can be found in the work by Arora–Crespo-Blanco–Winkert [5, Proposition 3.7].

Proposition 2.1. *Let hypotheses* (A1) *be satisfied. Then the following hold:*

(i) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ *is continuous;*
(ii) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *is continuous;*
(iii) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{r}(\Omega)$ *is compact for all* $1 \leq r < p^*$.

Moreover, the relation between the norm $\|\cdot\|$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and the modular function $\rho_{\mathcal{H}_{\log}}$ is stated in the next proposition, see Arora–Crespo-Blanco–Winkert [5, Proposition 3.6]. In the following, we denote by κ the constant given by

$$\kappa = \frac{e}{e+t_0}, \tag{2.1}$$

where e is Euler's number and t_0 is the positive number that satisfies $t_0 = e\log(e+t_0)$.

Proposition 2.2. *Let hypotheses* (A1) *be satisfied,* $\lambda > 0$, $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$, *and* κ *as in* (2.1). *Then the following hold:*

(i) $\|u\| = \lambda$ *if and only if* $\rho_{\mathcal{H}_{\log}}\left(\frac{\nabla u}{\lambda}\right) = 1$;
(ii) $\|u\| < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) *if* $\|u\| < 1$ *then* $\|u\|^{q+\kappa} \leq \rho_{\mathcal{H}_{\log}}(\nabla u) \leq \|u\|^{p}$;
(iv) *if* $\|u\| > 1$ *then* $\|u\|^{p} \leq \rho_{\mathcal{H}_{\log}}(\nabla u) \leq \|u\|^{q+\kappa}$;
(v) $\|u_n\| \to 0$ *if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u_n) \to 0$ *as* $n \to \infty$.

The following lemma will be used later, see Arora–Crespo-Blanco–Winkert [5, Lemma 5.4].

Lemma 2.3. *Let* $Q > 1$ *and* $h\colon [0,\infty) \to [0,\infty)$ *given by* $h(t) = \frac{t}{Q(e+t)\log(e+t)}$. *Then* h *attains its maximum value at* t_0 *and the value is* $\frac{\kappa}{Q}$, *where* t_0 *and* κ *are the same as in* (2.1).

Next, we introduce the nonlinear operator $A\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$. The properties of $A\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ are summarized in the following proposition, see Arora–Crespo-Blanco–Winkert [5, Theorem 4.4].

Theorem 2.4. *Let hypotheses* (A1) *be satisfied and* A *be given as in* (2.2). *Then* A *is bounded, continuous, strictly monotone, and satisfies the* (S_+)*-property, that is, any sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ *and* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$ *converges strongly to* u *in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

In the following, $C_0^1(\overline{\Omega})$ stands for the ordered Banach space given by

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega})\colon u\big|_{\partial\Omega} = 0\right\},$$

while $C_0^1(\overline{\Omega})_+$ is the positive cone defined by

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega})\colon u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\},$$

which has a nonempty interior

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega}) \colon u(x) > 0 \ \forall x \in \Omega \text{ and } \frac{\partial u}{\partial n}(x) < 0 \ \forall x \in \partial\Omega\right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$. For any $t \in \mathbb{R}$ we put $t_\pm = \max\{\pm t, 0\}$, that is, $t = t_+ - t_-$ and $|t| = t_+ + t_-$. Furthermore, for any function $u \colon \Omega \to \mathbb{R}$ we write $u_\pm(\cdot) = [u(\cdot)]_\pm$.

Let us now recall some known results about the eigenvalue problem of the p-Laplacian for $1 < p < \infty$ with homogeneous Dirichlet boundary condition which is defined by

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

We know from Lê [20] that there exists a smallest eigenvalue λ_1 of (2.3) which is positive, isolated, simple and can be written as

$$\lambda_1 = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \colon u \in W_0^{1,p}(\Omega), u \neq 0\right\}. \tag{2.4}$$

We denote by u_1 the L^p-normalized positive eigenfunction corresponding to λ_1, that is, $\|u_1\|_p = 1$. Furthermore, $u_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ due to the regularity theory of Lieberman [21] and the maximum principle by Pucci–Serrin [32].

Let X be a Banach space and X^* be its dual space. We say that a functional $\varphi \in C^1(X)$ satisfies the Palais-Smale condition (PS-condition for short), if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence. We also set

$$K_\varphi := \{u \in X \colon \varphi'(u) = 0\},$$

which is the set of all critical points of φ. Recall that a set $\mathcal{S} \subseteq X$ is called downward directed if for given $u_1, u_2 \in \mathcal{S}$ there exists $u \in \mathcal{S}$ such that $u \leq u_1$ and $u \leq u_2$. Similarly, $\mathcal{S} \subseteq X$ is called upward directed if for given $v_1, v_2 \in \mathcal{S}$ one can find $v \in \mathcal{S}$ such that $v_1 \leq v$ and $v_2 \leq v$.

The proof of Theorem 1.1 relies on the following abstract critical point result established by Kajikiya [19, Theorem 1], which extends the symmetric mountain pass theorem.

Theorem 2.5. *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and $\varphi \in C^1(X, \mathbb{R})$ such that the following hold:*

(i) *φ is even, bounded from below, $\varphi(0) = 0$ and it satisfies the* (PS)*-condition.*
(ii) *For any $n \in \mathbb{N}$, there exist a n-dimensional subspace X_n of X and a number $r_n > 0$ such that $\sup_{X_n \cap S_{r_n}} \varphi(u) < 0$, where $S_{r_n} = \{u \in X \colon \|u\| = r_n\}$.*

Then, the functional φ admits a sequence of critical points $\{v_n\}_{n\in\mathbb{N}}$ satisfying $\|v_n\| \to 0$ as $n \to \infty$.

3. Asymptotically vanishing sign-changing solutions

We first study a truncated auxiliary problem which helps us to deal with the critical term in (1.5). To this end, let $\Psi \in C^1(\mathbb{R})$ be an even cut-off function such that

$$\operatorname{supp}\Psi \subseteq [-\xi, \xi], \quad \Psi_{|\left[\frac{-\xi}{2}, \frac{\xi}{2}\right]} \equiv 1 \quad \text{and} \quad 0 < \Psi \leq 1 \quad \text{on } (-\xi, \xi). \tag{3.1}$$

Next, we introduce the function $\vartheta \colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\vartheta(x, s) = \Psi(s)\left[g(x, s) + |s|^{p^*-2}s\right] + (1 - \Psi(s))|s|^{\gamma-2}s, \tag{3.2}$$

which is a Carathéodory function, whereby γ is from hypothesis (A4). It is easy to see that from the choice of Ψ in (3.1) along with (3.2) and (A4) we have the growth condition

$$|\vartheta(x, s)| \leq C\left(1 + |s|^{\gamma-1}\right) \tag{3.3}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$ with some $C > 0$.

The strategy for dealing with the critical term in problem (1.5) relies on the cut-off function Ψ, introduced above and satisfying the properties listed in (3.1). In this framework, the function ϑ has subcritical growth (see (3.3)), and therefore, by considering the auxiliary problem formulated below, extremal constant sign solutions can be obtained through standard variational methods. Furthermore, Theorem 2.5 yields a sequence of sign-changing solutions w_n to the auxiliary problem converging to zero. This convergence makes it possible to select a sufficiently large $n_0 \in \mathbb{N}$ such that $\Psi(w_n(x)) = 1$ for a.a. $x \in \Omega$ and for all $n \geq n_0$ (by virtue of the second property in (3.1)), which in turn ensures that ϑ coincides with the right-hand side of the original problem (1.5). Note again that the number $\xi > 0$ is fixed from the beginning, see (A2).

Now, we are interested in the solvability of the auxiliary problem

$$-\operatorname{div}\mathcal{K}(u) = \vartheta(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{3.4}$$

where $\operatorname{div}\mathcal{K}(u)$ is the logarithmic double phase operator given in (1.2). We are going to prove the existence of extremal constant sign solutions of (3.4) which will be used in the construction of sign-changing solutions to our original problem (1.5). For this purpose, let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the sets of positive and negative solutions of problem (3.4), respectively. In the following, we denote by $\mathcal{E}_\pm \colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ the truncated energy functionals related to (3.4) given by

$$\mathcal{E}_\pm(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q \log(e + |\nabla u|)\right] \mathrm{d}x - \int_\Omega \Theta(x, \pm u_\pm)\, \mathrm{d}x, \tag{3.5}$$

for all $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$, where $\Theta(x, s) = \int_0^s \vartheta(x, t)\, \mathrm{d}t$. It is obvious to see that $\mathcal{E}_\pm \in C^1(W_0^{1,\mathcal{H}_{\log}}(\Omega))$.

First, we show that $\mathcal{S}_\pm$ are nonempty.

Proposition 3.1. *Let hypotheses* (A1)–(A5) *be satisfied. Then $\mathcal{S}_+$ and $\mathcal{S}_-$ are nonempty subsets in $W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$.*

Proof. We start by showing that $\mathcal{S}_+ \neq \emptyset$. Due to

$$\mathcal{E}_+(u) \geq \frac{1}{q}\rho_{\mathcal{H}_{\log}}(|\nabla u|) - \int_\Omega \Theta(x, u_+)\, \mathrm{d}x$$

along with the growth in (3.3), $\gamma < p$ by (A4) as well as Proposition 2.2 (iv), we see that $\mathcal{E}_+$ is coercive. Moreover, from Proposition 2.1 (iii), we know that $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for any $1 \leq r < p^*$. Therefore, the functional $\mathcal{E}_+$ is also sequentially weakly lower semicontinuous. Then, we can find an element $\hat{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\mathcal{E}_+(\hat{u}) = \inf \left[\mathcal{E}_+(u) \colon u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)\right].$$

We show that $\hat{u} \neq 0$. Taking hypothesis (A5) into account, for each $\varepsilon > 0$, there exists $\omega \in \left(0, \min\{\frac{\xi}{2}, 1\}\right)$ such that

$$G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t \geq \frac{\varepsilon}{p}|s|^p \quad \text{for all } |s| \leq \omega. \tag{3.6}$$

Recall that $u_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the L^p-normalized positive eigenfunction corresponding to λ_1 of the eigenvalue problem (2.3). Now we choose $t \in (0,1)$ small enough such that $tu_1(x) \in (0,\omega]$ for all $x \in \overline{\Omega}$. Then, since $\omega \in \left(0, \min\{\frac{\xi}{2}, 1\}\right)$, we get from (3.1) that

$$\vartheta(x, tu_1) = g(x, tu_1) + (tu_1)^{p^*-2} tu_1 \geq g(x, tu_1). \tag{3.7}$$

Now, using the representation of λ_1 in (2.4), $\|u_1\|_p = 1$ and the inequality $\log(e+xy) \leq \log(e+x) + \log(e+y)$ for all $x, y > 0$ as well as (3.6) and (3.7), we obtain

$$\begin{aligned}
\mathcal{E}_+(tu_1) &= \int_\Omega \left[\frac{1}{p}|\nabla(tu_1)|^p + \frac{\mu(x)}{q}|\nabla(tu_1)|^q \log(e + t|\nabla u_1|)\right] \mathrm{d}x \\
&\quad - \int_\Omega \Theta\left(x, tu_1\right) \mathrm{d}x \\
&\leq \frac{t^p}{p}\lambda_1 + \frac{t^q \log(e+t)}{q} \int_\Omega \mu(x)|\nabla u_1|^q \,\mathrm{d}x \\
&\quad + \frac{t^q}{q} \int_\Omega \mu(x)|\nabla u_1|^q \log(e + |\nabla u_1|)\,\mathrm{d}x - \frac{t^p}{p}\varepsilon \\
&= \frac{t^p}{p}(\lambda_1 - \varepsilon) + \frac{t^q \log(e+t)}{q} \int_\Omega \mu(x)|\nabla u_1|^q \,\mathrm{d}x \\
&\quad + \frac{t^q}{q} \int_\Omega \mu(x)|\nabla u_1|^q \log(e + |\nabla u_1|)\,\mathrm{d}x.
\end{aligned} \tag{3.8}$$

Taking $\varepsilon > \lambda_1$, we see from (3.8), for $t > 0$ sufficiently small, since $p < q$, that

$$\mathcal{E}_+(tu_1) < 0.$$

This shows that $\hat{u} \neq 0$.

Since $\hat{u}$ is a global minimizer of $\mathcal{E}_+$, it holds $\mathcal{E}_+'(\hat{u}) = 0$, which means

$$\begin{aligned}
&\int_\Omega \left(|\nabla \hat{u}|^{p-2}\nabla \hat{u} + \mu(x)\left(\log(e + |\nabla \hat{u}|) + \frac{|\nabla \hat{u}|}{q(e + |\nabla \hat{u}|)}\right)|\nabla \hat{u}|^{q-2}\nabla \hat{u}\right) \cdot \nabla \varphi \,\mathrm{d}x \\
&= \int_\Omega \vartheta(x, \hat{u}_+)\,\varphi \,\mathrm{d}x
\end{aligned}$$

for all $\varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Testing the above equation with $\varphi = -\hat{u}_- \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ (see Arora–Crespo-Blanco–Winkert [5, Proposition 3.8 (iii)]) yields $\hat{u}_- = 0$. Therefore, $\hat{u} \geq 0$ and since $\hat{u} \neq 0$, it is a nontrivial positive weak solution of problem (3.4). This proves $\mathcal{S}_+ \neq \emptyset$ and from Rădulescu–Stapenhorst–Winkert [33], we know that $\hat{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$.

Similarly, we can show the existence of a nontrivial negative bounded weak solution $\hat{v}$ of problem (3.4) which is the global minimizer of $\mathcal{E}_- \colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ defined in (3.5). □

In the next step, we will show that the auxiliary problem (3.4) has extremal constant sign solutions in the sense that there exist a smallest positive solution $\tilde{u} \in \mathcal{S}_+$ and a largest negative solution $\tilde{v} \in \mathcal{S}_-$.

Proposition 3.2. *Let hypotheses* (A1)–(A5) *be satisfied. Then there exists $\tilde{u} \in \mathcal{S}_+$ such that $\tilde{u} \leq u$ for all $u \in \mathcal{S}_+$ and there exists $\tilde{v} \in \mathcal{S}_-$ such that $\tilde{v} \geq v$ for all $v \in \mathcal{S}_-$.*

Proof. We start with the existence of $\tilde{u}$. First, note that the set $\mathcal{S}_+$ is downward directed. This is a standard proof and can be done as in the paper by Papageorgiou–Rădulescu–Repovš [28, Proposition 7]. From this fact, using Lemma 3.10 by Hu–Papageorgiou [18], there exists a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf \mathcal{S}_+.$$

As $u_n \in \mathcal{S}_+$ it holds

$$\begin{aligned}
&\int_\Omega \Bigg(|\nabla u_n|^{p-2}\nabla u_n \\
&\qquad + \mu(x)\left(\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right)|\nabla u_n|^{q-2}\nabla u_n \Bigg)\cdot \nabla\varphi \,\mathrm{d}x \\
&= \int_\Omega \vartheta(x,u_n)\varphi\,\mathrm{d}x
\end{aligned} \tag{3.9}$$

for all $\varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and for all $n \in \mathbb{N}$. Choosing $\varphi = u_n \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ in (3.9) and using (3.3) as well as $0 \leq u_n \leq u_1$ leads to

$$\rho_{\mathcal{H}_{\log}}(\nabla u_n) = \int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^q \log(e+|\nabla u_n|)\,\mathrm{d}x < c_1$$

for some $c_1 > 0$ and for all $n \in \mathbb{N}$. Therefore, Proposition 2.2 (iii), (iv) implies that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is bounded. Furthermore, taking hypothesis (A4) into account, we see that $\gamma < \frac{p^2}{N-p} + 1$ and so $\frac{N}{p}(\gamma - 1) < p^*$. Next, we can take a number $t > \frac{N}{p}$ such that $t(\gamma-1) < p^*$. Then, by the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and Proposition 2.1 (iii) we may assume that

$$u_n \rightharpoonup \tilde{u} \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad u_n \to \tilde{u} \quad \text{in } L^{t(\gamma-1)}(\Omega) \tag{3.10}$$

for a subsequence if necessary (not relabeled) and $\tilde{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Moreover, combining (3.1), (3.2) and hypothesis (A4) we get

$$|\vartheta(x,s)| \leq c_2|s|^{\gamma-1} \tag{3.11}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for some $c_2 > 0$. Now, from (3.9) and (3.11), because of $t > \frac{N}{p}$, we obtain that

$$\|u_n\|_\infty \le c_3 \|u_n\|_{t(\gamma-1)}^{\frac{\gamma-1}{p-1}} \tag{3.12}$$

for some $c_3 > 0$ and for all $n \in \mathbb{N}$. The proof of this result can be done as in Perera–Squassina [31, Proposition 2.4] since $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ by Proposition 2.1 (i).

We are going to prove that $\tilde{u} \neq 0$. Suppose by contradiction that $\tilde{u} = 0$. Then from (3.10) and (3.12) we have $\|u_n\|_\infty \to 0$ as $n \to +\infty$ which implies the existence of $n_0 \in \mathbb{N}$ such that $0 < u_n(x) \le \omega$ for a.a. $x \in \Omega$ and for all $n \ge n_0$, where $\omega \in \big(0, \min\{\frac{\xi}{2}, 1\}\big)$. Hence, taking (3.1) and (3.2) into account yields

$$\vartheta(x, u_n(x)) = g(x, u_n(x)) + u_n(x)^{p^*-1} \tag{3.13}$$

for a.a. $x \in \Omega$ and for all $n \ge n_0$. Next, we set $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$ which gives $\|y_n\| = 1$ and $y_n \ge 0$ for all $n \in \mathbb{N}$. Therefore, we may assume, for a subsequence if necessary (not relabeled), that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega)$$

for $y \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ with $y \ge 0$. Applying $u_n = \|u_n\| y_n$ in (3.9) and using (3.13) results in

$$\begin{aligned}
&\int_\Omega \bigg(\|u_n\|^{p-1} |\nabla y_n|^{p-2} \nabla y_n \\
&\qquad + \mu(x) \|u_n\|^{q-1} \left(\log(e + |\nabla u_n|) + \frac{|\nabla u_n|}{q(e + |\nabla u_n|)} \right) |\nabla y_n|^{q-2} \nabla y_n \bigg) \cdot \nabla \varphi \, \mathrm{d}x \\
&= \int_\Omega \|u_n\|^{p-1} \left[\frac{g(x, u_n)}{u_n^{p-1}} + u_n^{p^*-p} \right] y_n^{p-1} \varphi \, \mathrm{d}x
\end{aligned}$$

for all $\varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and for all $n \ge n_0$. From this we conclude that

$$\begin{aligned}
&\int_\Omega \bigg(|\nabla y_n|^{p-2} \nabla y_n \\
&+ \mu(x) \|u_n\|^{q-p} \left(\log(e + |\nabla u_n|) + \frac{|\nabla u_n|}{q(e + |\nabla u_n|)} \right) |\nabla y_n|^{q-2} \nabla y_n \bigg) \cdot \nabla \varphi \, \mathrm{d}x \\
&= \int_\Omega \left[\frac{g(x, u_n)}{u_n^{p-1}} + u_n^{p^*-p} \right] y_n^{p-1} \varphi \, \mathrm{d}x
\end{aligned} \tag{3.14}$$

for all $\varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and for all $n \ge n_0$. Note that

$$\begin{aligned}
\log(e + |\nabla u_n|) &= \log(e + \|u_n\| |\nabla y_n|) \\
&\le \begin{cases} \log(e + |\nabla y_n|) & \text{if } \|u_n\| < 1, \\ \|u_n\| \log(e + |\nabla y_n|) & \text{if } \|u_n\| \ge 1, \end{cases}
\end{aligned} \tag{3.15}$$

where we used in case $\|u_n\| < 1$ the monotonicity of the logarithmic function while for $\|u_n\| \ge 1$ the standard inequality $\log(e + Ct) \le C \log(e + t)$ for all $t \ge 0$ and $C \ge 1$. Therefore, using (3.15) and Lemma 2.3, we see that the left-hand side of (3.14) is bounded for all $\varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ (similar to the proof of Theorem 4.4 by Arora–Crespo-Blanco–Winkert [5]) and so the same holds for the right-hand side of (3.14). But then, using (A5), we see that

$$y=0 \quad \text{and} \quad \frac{g(x,u_n(x))}{u_n(x)^{p-1}}y_n(x)^{p-1} \to 0 \quad \text{for a.a. } x \in \Omega.$$

Next, choosing $\varphi = y_n$ in (3.14) and passing to the limit as $n \to +\infty$, we arrive at

$$\lim_{n\to+\infty} \int_\Omega |\nabla y_n|^p \,\mathrm{d}x = 0.$$

Therefore, we have $\nabla y_n(x) \to 0$ for a.a. $x \in \Omega$ for a subsequence if necessary, not relabeled. This implies that $\mathcal{H}_{\log}(x, \nabla y_n) \to 0$ for a.a. $x \in \Omega$. From Vitali's convergence theorem we know that $\{\mathcal{H}_{\log}(\,\cdot\,, \nabla y_n(\cdot)\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ is uniformly integrable which yields

$$\rho_{\mathcal{H}_{\log}}(\nabla y_n) \to 0 \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega). \tag{3.16}$$

Recall that by construction we have $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Taking Proposition 2.2 (ii) into account, this is equivalent to $\rho_{\mathcal{H}_{\log}}(\nabla y_n) = 1$ for all $n \in \mathbb{N}$ which is a contradiction to (3.16). Thus, we have $\tilde{u} \neq 0$ and $\tilde{u} \in \mathcal{S}_+$ is the smallest positive solution of (3.4) in $\mathcal{S}_+$. Using similar arguments, one can prove that $\tilde{v} \in \mathcal{S}_-$ such that $\tilde{v} = \sup \mathcal{S}_-$. □

Remark 3.3. By definition, $g(x,\cdot)$ is defined only locally. Then, because of hypothesis (A5), namely

$$\lim_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

without any loss of generality, we can assume that

$$\frac{g(x,s)}{|s|^{p-2}s} > 0 \qquad \text{for a.a. } x \in \Omega \text{ and for all } |s| \le \xi.$$

This implies

$$g(x,s) > 0 \quad \text{for all } 0 < s \le \xi \quad \text{and} \quad g(x,s) < 0 \quad \text{for all } -\xi \le s < 0.$$

Let

$$[\tilde{v}, \tilde{u}] := \left\{ u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \colon \tilde{v}(x) \le u(x) \le \tilde{u}(x) \text{ for a.a. } x \in \Omega \right\},$$

where $\tilde{u}$ and $\tilde{v}$ are the extremal constant sign solutions from Proposition 3.2. Next, we introduce the cut-off function $\tilde{\vartheta} \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{\vartheta}(x,s) := \begin{cases} \vartheta(x,\tilde{v}(x)) & \text{if } s < \tilde{v}(x), \\ \vartheta(x,s) & \text{if } \tilde{v}(x) \le s \le \tilde{u}(x), \\ \vartheta(x,\tilde{u}(x)) & \text{if } \tilde{u}(x) < s \end{cases} \tag{3.17}$$

and consider the truncated C^1-functional $\tilde{\mathcal{E}} \colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ by

$$\tilde{\mathcal{E}}(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q \log(e+|\nabla u|) \right] \mathrm{d}x - \int_\Omega \tilde{\Theta}(x,u)\,\mathrm{d}x,$$

for all $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$, where $\tilde{\Theta}(x,s) = \int_0^s \tilde{\vartheta}(x,t)\,\mathrm{d}t$.

Note that $K_{\tilde{\mathcal{E}}} = \{u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)\colon (\tilde{\mathcal{E}})'(u) = 0\} \subseteq [\tilde{v}, \tilde{u}]$. Indeed, taking $u \in K_{\tilde{\mathcal{E}}} \setminus \{\tilde{u}, \tilde{v}\}$ gives

$$\begin{aligned}&\int_\Omega \Big(|\nabla u|^{p-2}\nabla u\\ &+ \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\Big)\cdot\nabla\varphi \,\mathrm{d}x\\ &= \int_\Omega \tilde{\vartheta}(x,u)\varphi\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,\mathcal{H}_{\log}}(\Omega).\end{aligned} \tag{3.18}$$

Testing (3.18) with $\varphi = (u-\tilde{u})_+ \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and using that $\tilde{u}$ solves (3.4), we obtain

$$\begin{aligned}&\langle A(u), (u-\tilde{u})_+\rangle\\ &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-\tilde{u})_+\,\mathrm{d}x\\ &\quad + \int_\Omega \mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\cdot\nabla(u-\tilde{u})_+\,\mathrm{d}x\\ &= \int_\Omega \tilde{\vartheta}(x,u)(u-\tilde{u})_+\,\mathrm{d}x\\ &= \int_\Omega \vartheta(x,\tilde{u})(u-\tilde{u})_+\,\mathrm{d}x\\ &= \int_\Omega |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\cdot\nabla(u-\tilde{u})_+\,\mathrm{d}x\\ &\quad + \int_\Omega \mu(x)\left(\log(e+|\nabla \tilde{u}|)+\frac{|\nabla \tilde{u}|}{q(e+|\nabla \tilde{u}|)}\right)|\nabla \tilde{u}|^{q-2}\nabla \tilde{u}\cdot\nabla(u-\tilde{u})_+\,\mathrm{d}x\\ &= \langle A(\tilde{u}), (u-\tilde{u})_+\rangle.\end{aligned}$$

Therefore,

$$\langle A(u) - A(\tilde{u}), (u-\tilde{u})_+\rangle = 0,$$

which implies, due to the strict monotonicity of A (see Proposition 2.4), that $u \le \tilde{u}$. Testing (3.18) with $\varphi = (\tilde{v} - u)_+$ and reasoning as above shows that $\tilde{v} \le u$. Thus, it holds $K_{\tilde{\mathcal{E}}} \subseteq [\tilde{v}, \tilde{u}]$.

Now, let $V \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ be a finite dimensional subspace.

Proposition 3.4. *Let hypotheses* (A1)–(A5) *be satisfied. Then, there exists a number* $h_V > 0$ *such that*

$$\sup\left[\tilde{\mathcal{E}}(v)\colon v \in V,\ \|v\| = h_V\right] < 0.$$

Proof. Recall that all norms on V are equivalent since V is finite dimensional (see Papageorgiou–Winkert [30, Proposition 3.1.17]). Then we can find $h_V > 0$ such that

$$v \in V \quad \text{and} \quad \|v\| \le h_V \quad \text{imply} \quad |v(x)| \le \omega \quad \text{for a.a. } x \in \Omega,$$

where $\omega \in \left(0, \min\{\frac{\xi}{2}, 1\}\right)$ is as in the proof of Proposition 3.1. Since $\omega < \frac{\xi}{2}$, by (3.1), we have $\Psi(v(x)) = 1$ for a.a. $x \in \Omega$. From this, $v \in V$ with $\|v\| \le h_V$, we see that

$$\tilde{\vartheta}(x, v(x)) = \begin{cases} g(x, \tilde{v}(x)) + |\tilde{v}(x)|^{p^*-2}\tilde{v}(x) & \text{if } v(x) < \tilde{v}(x), \\ g(x, v(x)) + |v(x)|^{p^*-2}v(x) & \text{if } \tilde{v}(x) \le v(x) \le \tilde{u}(x), \\ g(x, \tilde{u}(x)) + |\tilde{u}(x)|^{p^*-2}\tilde{u}(x) & \text{if } \tilde{u}(x) < v(x). \end{cases}$$

Let $\tilde{g}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\tilde{g}(x, v(x)) = \begin{cases} g(x, \tilde{v}(x)) & \text{if } v(x) < \tilde{v}(x), \\ g(x, v(x)) & \text{if } \tilde{v}(x) \le v(x) \le \tilde{u}(x), \\ g(x, \tilde{u}(x)) & \text{if } \tilde{u}(x) < v(x). \end{cases}$$

For $\tilde{G}(x, s) := \int_0^s \tilde{g}(x, t)\,\mathrm{d}t$ and $v < \tilde{v}$ we have

$$\begin{aligned} \tilde{G}(x, v) &= \int_0^{\tilde{v}} \tilde{g}(x, s)\,\mathrm{d}s + \int_{\tilde{v}}^{v} \tilde{g}(x, s)\,\mathrm{d}s = \int_0^{\tilde{v}} g(x, s)\,\mathrm{d}s + \int_{\tilde{v}}^{v} g(x, \tilde{v})\,\mathrm{d}s \\ &= G(x, \tilde{v}) + g(x, \tilde{v})(v - \tilde{v}), \end{aligned}$$

where $G(x, s) = \int_0^t g(x, t)\,\mathrm{d}t$. By Remark 3.3, we know that $g(x, \tilde{v}) < 0$ for a.a. $x \in \Omega$. Then it follows $g(x, \tilde{v})(v - \tilde{v}) > 0$ for a.a. $x \in \Omega$ and so

$$\begin{aligned} G(x, v) - \tilde{G}(x, v) &= G(x, v) - G(x, \tilde{v}) + g(x, \tilde{v})(\tilde{v} - v) \\ &\le G(x, v) - G(x, \tilde{v}). \end{aligned}$$

Arguing in the same way, for $\tilde{u} < v$ it holds

$$\tilde{G}(x, v) = G(x, \tilde{u}) + g(x, \tilde{u})(v - \tilde{u}),$$

and so, since $g(x, \tilde{u})(\tilde{u} - v) < 0$ by Remark 3.3,

$$\begin{aligned} G(x, v) - \tilde{G}(x, v) &= G(x, v) - G(x, \tilde{u}) + g(x, \tilde{u})(\tilde{u} - v) \\ &\le G(x, v) - G(x, \tilde{u}). \end{aligned}$$

On account of this, we can write

$$\begin{aligned} \tilde{\mathcal{E}}(v) &= \int_\Omega \left[\frac{1}{p}|\nabla v|^p + \frac{\mu(x)}{q}|\nabla v|^q \log(e + |\nabla v|)\right] \mathrm{d}x - \int_\Omega \tilde{\Theta}(x, v)\,\mathrm{d}x \\ &= \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e + |\nabla v|)\,\mathrm{d}x \\ &\quad - \int_{\{x \in \Omega\colon\, v(x) < \tilde{v}(x)\}} \left(\tilde{G}(x, v) + \frac{1}{p^*}|\tilde{v}|^{p^*}\right) \mathrm{d}x \\ &\quad - \int_{\{x \in \Omega\colon\, \tilde{v}(x) \le v(x) \le \tilde{u}(x)\}} \left[G(x, v) + \frac{1}{p^*}|v|^{p^*}\right) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&- \int\limits_{\{x\in\Omega:\ \tilde{u}(x)<v(x)\}} \left(\tilde{G}(x,v) + \frac{1}{p^*}|\tilde{u}|^{p^*}\right) \mathrm{d}x \\
&\leq \frac{1}{p}\int\limits_{\Omega} |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int\limits_{\Omega} \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \\
&- \int\limits_{\{x\in\Omega:\ v(x)<\tilde{v}(x)\}} \tilde{G}(x,v)\,\mathrm{d}x \\
&- \int\limits_{\{x\in\Omega:\ \tilde{v}(x)\leq v(x)\leq \tilde{u}(x)\}} G(x,v)\,\mathrm{d}x \\
&- \int\limits_{\{x\in\Omega:\ \tilde{u}(x)<v(x)\}} \tilde{G}(x,v)\,\mathrm{d}x \\
&= \frac{1}{p}\int\limits_{\Omega} |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int\limits_{\Omega} \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x - \int\limits_{\Omega} G(x,v)\,\mathrm{d}x \\
&+ \int\limits_{\{x\in\Omega:\ v(x)<\tilde{v}(x)\}} \left(G(x,v)-\tilde{G}(x,v)\right) \mathrm{d}x \\
&+ \int\limits_{\{x\in\Omega:\ \tilde{u}(x)<v(x)\}} \left(G(x,v)-\tilde{G}(x,v)\right) \mathrm{d}x \\
&\leq \frac{1}{p}\int\limits_{\Omega} |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int\limits_{\Omega} \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x - \int\limits_{\Omega} G(x,v)\,\mathrm{d}x \\
&+ \int\limits_{\{x\in\Omega:\ v(x)<\tilde{v}(x)\}} \left(G(x,v)-G(x,\tilde{v})\right) \mathrm{d}x \\
&+ \int\limits_{\{x\in\Omega:\ \tilde{u}(x)<v(x)\}} \left(G(x,v)-G(x,\tilde{u})\right) \mathrm{d}x.
\end{aligned}
$$

Recall (3.6), that is, by (A5), for each $\varepsilon > 0$, we can find $\omega \in \left(0, \min\{\frac{\xi}{2}, 1\}\right)$ such that

$$
G(x,s) \geq \frac{\varepsilon}{p}|s|^p \quad \text{for all } |s| \leq \omega. \tag{3.19}
$$

Now we can choose $h_V > 0$ sufficiently small such that

$$
\begin{aligned}
&\int\limits_{\{x\in\Omega:\ v(x)<\tilde{v}(x)\}} \left(G(x,v)-G(x,\tilde{v})\right) \mathrm{d}x \\
&+ \int\limits_{\{x\in\Omega:\ \tilde{u}(x)<v(x)\}} \left(G(x,v)-G(x,\tilde{u})\right) \mathrm{d}x < \omega^p.
\end{aligned} \tag{3.20}
$$

Using (3.19) and (3.20) in the observations above we obtain

$$\tilde{\mathcal{E}}(v) \le \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \\ - \frac{\varepsilon}{p}\int_\Omega |v|^p\,\mathrm{d}x + \omega^p. \tag{3.21}$$

From Proposition 2.2 (iii), (iv) we know that

$$\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \le \rho_{\mathcal{H}_{\log}}(\nabla v) \le \max\{\|v\|^p,\ \|v\|^{q+\kappa}\}. \tag{3.22}$$

Recall that all norms on V are equivalent since V is finite dimensional. Using this fact and (3.22) in (3.21) we can find positive constants c_1, c_2, c_3, independent of ω, such that

$$\tilde{\mathcal{E}}(v) \le c_1\|v\|_\infty^p + c_2\max\{\|v\|_\infty^p,\ \|v\|_\infty^{q+\kappa}\} - \varepsilon c_3\|v\|_\infty^p + \omega^p.$$

Then, for $v \in V$ with $\|v\| = h_V$ along with the equivalence of the norms on V, it follows, due to $\omega < 1$, that

$$\begin{aligned}\tilde{\mathcal{E}}(v) &\le c_1\omega^p + c_2\max\{\omega^p, \omega^{q+\kappa}\} - \varepsilon c_3\omega^p + \omega^p\\ &= (c_1 + c_2 - \varepsilon c_3 + 1)\,\omega^p.\end{aligned}$$

Choosing $\varepsilon > \frac{c_1+c_2+1}{c_3}$ yields $\tilde{\mathcal{E}}(v) < 0$ for all $v \in V$ with $\|v\| = h_V$. $\square$

Now we are in the position to prove Theorem 1.1 by applying the symmetric mountain pass theorem due to Kajikiya [19, Theorem 1].

Proof of Theorem 1.1. First note that the truncated functional $\tilde{\mathcal{E}}\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ is even and coercive. In particular, it is bounded from below. Then, from Proposition 5.1.15 by Papageorgiou–Rădulescu–Repovš [27] we know that it fulfills the PS-condition. Using this and Proposition 3.4 we are able to apply Theorem 2.5 to get a sequence $\{w_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ such that

$$w_n \in K_{\tilde{\mathcal{E}}} \subseteq [\tilde{v}, \tilde{u}], \quad w_n \neq 0, \quad \tilde{\mathcal{E}}(w_n) \le 0 \quad \text{for all } n \in \mathbb{N}$$

and

$$\|w_n\| \to 0 \quad \text{as } n \to +\infty. \tag{3.23}$$

Recall that the functions $\tilde{v}$ and $\tilde{u}$ are the extremal constant sign solutions of (3.4), see Proposition 3.2. Since $w_n \in K_{\tilde{\mathcal{E}}} \subseteq [\tilde{v}, \tilde{u}]$ and $w_n \neq 0$ for all $n \in \mathbb{N}$, we know that w_n is a critical point of $\tilde{\mathcal{E}}$ belonging to $[\tilde{v}, \tilde{u}]$. Then, due to the truncation defined in (3.17), it follows that $\tilde{\vartheta}(x, s) = \vartheta(x, s)$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$. Therefore, w_n is a solution of our auxiliary problem (3.4) and since $w_n \in [\tilde{v}, \tilde{u}]$ with $\tilde{v}$, $\tilde{u}$ being the extremal constant sign solutions of (3.4), w_n must be a sign-changing solution of problem (3.4) for all $n \in \mathbb{N}$. Furthermore, as already pointed out in (3.12), we have the estimate

$$\|w_n\|_\infty \le C\|w_n\|_{t(\gamma-1)}^{\frac{\gamma-1}{p-1}}$$

for some $C > 0$ and for all $n \in \mathbb{N}$ with $t > \frac{N}{p}$ and $t(\gamma - 1) < p^*$. Then, due to (3.23), we obtain $\|w_n\|_\infty \to 0$ as $n \to +\infty$. In addition, there exists a number $n_0 \in \mathbb{N}$ such that $|w_n(x)| \le \frac{\xi}{2}$ for a.a. $x \in \Omega$ and for all $n \ge n_0$. From this we deduce that $\Psi(w_n(x)) = 1$ for a.a. $x \in \Omega$ and for all $n \ge n_0$, see (3.1). From this and (3.2) we see that w_n is a sign-changing solution of problem (1.5) for all $n > n_0$. $\square$

Ethical approval

Not applicable.

Authors' contributions

The authors contributed equally to this work.

Declaration of competing interest

The authors declare that they have no competing interests.

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