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SIMULATION OF AN AGE-STRUCTURED OPTIMAL HARVEST CONTROL MODEL

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Abstract. This paper investigates an age-structured optimal harvest control model from both theoretical and numerical perspectives. First, we apply classical optimal control techniques to establish the existence of an optimal control, derive the necessary conditions, conduct a steady-state analysis, and characterize the bang-bang regime. Subsequently, we numerically solve of the problem using an appropriate discretization method to support the theoretical results.

Key Words and Phrases: Numerical Simulation, Lotka-Mckendrick Model, Age-Structured Population, Optimal Control.

2010 Mathematics Subject Classifications: 34A30, 35M10, 35M11, 35M12, 65D25, 65D30. 65D32, 65D40, 65G20, 68R01, 76D05

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1. Introduction

The development and interaction of biological species are key areas of modern research. Mathematical modeling plays a crucial role in understanding population dynamics, offering reliable forecasts and strategic guidance for sustainable management. Although a wide range of mathematical models and analytical tools have been developed to study biological populations, practical challenges [4, 12] continue to demand new models [1, 5] that incorporate both exogenous and endogenous factors. Age is one of the most critical parameters influencing the structure of a population. Building upon the foundational work of Sharpe, Lotka, and McKendrick, structured population models based on partial differential equations have proven particularly effective. Age- or size-structured models with optimal control are especially valuable for managing resources and preserving species. In this paper, we analyze a model that describes the age density dynamics of a population subject to mortality and harvesting, excluding natural reproduction. The objective is to maximize profit while ensuring the population's sustainability. We begin with a theoretical analysis, presented in the section "Mathematical Analysis of the Model," followed by a numerical resolution using the finite difference method, and conclude with simulations performed in MATLAB.

2. Mathematical analysis of the model

Mathematical analysis plays an important role in understanding and validating the behavior of dynamic models. The primary objective of this section is to examine the fundamental properties of the proposed model—specifically, the existence, uniqueness, and regularity of solutions—while emphasizing the optimal control aspect.
Throughout this section, we consider $T \leq \infty$, $A < \infty$, and define the domain $Q_T =]0, T[\times]0, A[$.

2.1. Model presentation

Let $\mu(t,\tau)$ denote the mortality rate of individuals at time t and age τ. The governing equation for the age-structured population is derived from the principle:

$$\frac{\partial x(t,\tau)}{\partial t}+\frac{\partial x(t,\tau)}{\partial \tau}=-\mu(t,\tau)x(t,\tau) \tag{1}$$

where $x(t,\tau)$ is the population density. We aim to maximize the objective functional:

$$I=\int_0^{\infty} e^{-rt}\left(\int_0^A c(t,\tau)u(t,\tau)\,d\tau-k(t)p(t)\right)dt \tag{2}$$

subject to the constraints:

$$\begin{cases} \frac{\partial x(t,\tau)}{\partial t}+\frac{\partial x(t,\tau)}{\partial \tau}=-\mu(t,\tau)x(t,\tau)-u(t,\tau) & \text{in } Q_T \\ x(t,0)=p(t) & \text{in } [0,T[\\ x(0,\tau)=x_0(\tau) & \text{in } [0,A[\\ x(t,\tau)\geq 0 & \\ 0\leq p(t)\leq p_{max} & \\ 0\leq u(t,\tau)\leq u_{max}. & \end{cases} \tag{3}$$

- t is time and τ is age;
- I is the objective function or criterion function which is the discounted profit over the infinite horizon;
- $c(t,\tau)$ is the unit price of the crop on the market;
- $k(t)$ is the unit cost of introducing a new individual into the population;
- r is the discount rate;
- $x(t,\tau)$ is the population density depending on time t and age τ;
- $u(t,\tau)$ is the harvest rate considered as the unknown decision variable;
- $p(t)$ represents the new individuals introduced at age 0 which is an endogenous control;
- $x_0(\tau)$ is the initial condition at time $t=0$.

Functional framework

Before we begin solving the system, we assume some hypotheses.

H1 $\mu\in L^1(Q_T), \mu(t,\tau)\geq 0, \forall(t,\tau)\in Q_T$;
H2 $x_0\in L^{\infty}(]0,A[), x_0(\tau)\geq 0, \forall\tau\in]0,A[$;
H3 $p\in L\infty([0,T[), u\in L^{\infty}(Q_T)$.

Using the technique of characteristics we can find the integral equations of the solution of the system (2) which are given by:

$$x(t,\tau)=x_0(\tau-t)e^{-\int_0^t \mu(s,s+\tau-t)ds}-\int_0^t u(s,s+\tau-t)e^{-\int_s^t \mu(\sigma,\sigma+\tau-t)d\sigma}ds \quad \text{if} \quad t<\tau \tag{4}$$

$$x(t,\tau)=p(t-\tau)e^{-\int_0^{\tau} \mu(s+t-\tau,s)ds}-\int_0^{\tau} u(s+t-\tau,s)e^{-\int_s^{\tau} \mu(\sigma+t-\tau,\sigma)d\sigma}ds \quad \text{if} \quad t\geq\tau. \tag{5}$$

We can notice that the solution x to be sought, belongs to $L^{\infty}(Q_T)$.

Proposition 2.1.1 :
Under the assumptions $H1 - H3$, if p satisfies the Volterra integral equation then relations (4) and (5) can be written as an integral equation and the solution of this integral equation is the unique solution of the system (2).

Where a Volterra integral equation is given by:

$$x(t) = \int_0^t K(t-s)x(s)ds + f(t) \tag{6}$$

The proof of this proposition is based on the use of the fixed point theorem, one can consult [5] for more details.

Theorem 2.1.1 :
If the mortality rate satisfies

H4 $\quad \displaystyle\int_0^A \mu(t - A + \tau, \tau)d\tau = +\infty$ For all $t \in [0, T[$, where μ is extended by 0 on $]-\infty, 0[\times[0, A[$,

then the solution x of the system (2) satisfies

$$\lim_{\epsilon \to 0} x(t-\epsilon, A-\epsilon) = 0 \text{ for all } t \in [0, T[, \tag{7}$$

that is, $x(t, A) = 0$ for all $t \in [0, T[$.

Proof :
Using the expressions given by relations (4) and (6), we arrive at the justification of the theorem.

Considering $K = U \times P$ with

$$U = \{u \in L^\infty(Q_T), \quad 0 \leq u(t, \tau) \leq u_{\max}, (t, \tau) \in Q_T\}$$

and

$$P = \{p \in L^\infty(0, T), \quad 0 \leq p(t) \leq u_{\max}, \ t \in [0, T[\}$$

as the set of controls, our problem can be put in the form, find $u^*(t, \tau)$ and $p^*(t)$ such that:

$$I(u^*, p^*) = \max_{(u,p) \in K} I(u, p). \tag{8}$$

2.2. Existence results

The existence of the solution to the problem within the framework of age-structured optimal harvest control are fundamental aspects to ensure that the optimal strategy obtained is well defined..

Definition 2.2.1 :

1. A pair (u, p) solution of (8) is called optimal control.
2. The functions u and p are admissible if they satisfy the constraints

$$0 \leq u(t, \tau) \leq u_{max} \text{ and } 0 \leq p(t) \leq p_{max}$$

respectively.

3. The variations $\delta u(t,\tau)$ and $\delta p(t)$, $t \in [0,T[$ of $u(t,\tau)$ and $p(t)$, are admissible if $u(t,\tau)+\delta u(t,\tau)$ and $p(t)+\delta p(t)$, $t \in [0,T[$ are admissible as soon as $u(t,\tau)$ and $p(t)$, $t \in [0,T[$ are admissible.

Theorem 2.2.1 :
Suppose that I is continuous, K is a non-empty closed subset of $\mathbb{R}^n$ and that one of the following conditions is met:

1. and K be bounded,
2. let I be coercive.

Then the problem $(\mathcal{P})$ admits at least one solution [5].

As far as we are concerned, the functional I is linear and therefore continuous, since the dimension of the control space is finite and the set K is a compact set.
Hence the following proposition.

Proposition 2.2.1 :
Problem (8) admits at least one optimal control.

Proof :
We define the function Φ by:

$$\Phi(u,p) = \int_0^{\infty} e^{-rt}\left(\int_0^A c(t,\tau)u(t,\tau)d\tau - k(t)p(t)\right)dt$$

and let

$$d = \sup_{(u,p)\in\mathbb{K}} \Phi(u,p)$$

Using the constraints imposed by the system (2.2) on the controls, we have the following inequality:

$$0 \le \Phi(u,p) \le \int_0^{\infty} e^{-rt}\left(\int_0^A c(t,\tau)u_{\max}d\tau\right)dt$$

Since $\displaystyle\int_0^A c(t,\tau)d\tau$ converges i.e. $\displaystyle\int_0^A c(t,\tau)d\tau < \infty$.
So, we have

$$0 \le \Phi(u,p) \le \int_0^{\infty} e^{-rt}\left(\int_0^A c(t,\tau)u_{\max}d\tau\right)dt < \infty$$

Therefore $d \in [0,+\infty[$.
Consider for all $n \in \mathbb{N}^*$, $U_n = (u_n,p_n)$ a maximizing sequence of I on K and

$$d - \frac{1}{n} < \Phi(U_n) \le d.$$

.
The sequence $U_n = (u_n,p_n)$ is bounded on K. Since U_n is bounded on K there exists a subsequence denoted (again U_n) which converges to $\overline{U} = (\overline{u},\overline{p})$ in K which is closed.
Passing to the limit when $n \to \infty$ in

$$d - \frac{1}{n} < \Phi(U_n) \le d,$$

we obtain that:

$$d = \Phi(\overline{U})$$

that is to say that $\overline{U} = (\overline{u}, \overline{p})$ is an optimal control pair for problem (8).

2.3. Maximum principle

The strength of the maximum principle is to reduce an initially complex problem (maximizing a functional) to an optimization problem.
We use the maximum principle to provide the necessary conditions of optimality for our optimal control problem.

Proposition 2.3.1 :
If (u^*, p^*) is a solution to problem (8), then:

$$\begin{cases} \frac{\partial I}{\partial u} \leq 0 & \text{at} \quad u^*(t,\tau) = 0, \\ \frac{\partial I}{\partial u} \geq 0 & \text{at} \quad u^*(t,\tau) = u_{\max}, \\ \frac{\partial I}{\partial u} = 0 & \text{at} \quad 0 < u^*(t,\tau) < u_{\max}, \\ \frac{\partial I}{\partial p} \leq 0 & \text{at} \quad p^*(t) = 0, \\ \frac{\partial I}{\partial p} \geq 0 & \text{at} \quad p^*(t) = p_{\max}, \\ \frac{\partial I}{\partial p} = 0 & \text{at} \quad 0 < p^*(t) < p_{\max}. \end{cases} \tag{9}$$

where

$$\frac{\partial I}{\partial u} = e^{-rt}(c(t,\tau) - \lambda(t,\tau)), \tag{10}$$

$$\frac{\partial I}{\partial p} = e^{-rt}(\lambda(t,0) - k(t)), \tag{11}$$

$$\frac{\lambda(t,\tau)}{\partial t} + \frac{\partial \lambda(t,\tau)}{\partial \tau} = (r + \mu(t,\tau))\lambda(t,\tau) - \eta(t,\tau), \tag{12}$$

$$\lim_{t\to\infty} e^{-rt}\lambda(t,\tau) = 0,\ \tau \in [0, A[,\ \lambda(t,A) = 0,\ t \in [0,\infty[, \tag{13}$$

$$\eta(t,\tau) > 0 \text{ at } x^*(t,\tau) = 0,\ \eta(t,\tau) = 0 \text{ at } x^*(t,\tau) > 0. \tag{14}$$

the functions λ and η are adjoint variables associated with the constraints of the state variable x.

Proof :
The proof of this proposition is based on the use of the Lagrange multiplier method.
The Lagrangian of the problem is given by:

$$\begin{aligned} &L\left(x,u,p,\lambda,\eta\right) = I\left(u,p\right) - \\ &- \int_0^\infty e^{-rt} \int_0^A \lambda\left(t,\tau\right) \left(\tfrac{\partial x(t,\tau)}{\partial t} + \tfrac{\partial x(t,\tau)}{\partial \tau} + \mu\left(t,\tau\right) + u\left(t,\tau\right)\right) d\tau dt + \\ &+ \int_0^\infty e^{-rt} \int_0^A \eta\left(t,\tau\right) x\left(t,\tau\right) d\tau dt. \end{aligned} \tag{15}$$

The variation of the Lagrangian $\delta\mathcal{L}(u,p) = \mathcal{L}(u+\delta u, p+\delta p) - \mathcal{L}(u,p)$ is given by:

$$\delta\mathcal{L}(u,p) = \int_0^\infty e^{-rt} \left(\int_0^A c(t,\tau)\delta u - k(t)\delta p(t) \right) dt$$

$$- \int_0^\infty e^{-rt} \int_0^A \lambda(t,\tau)\Big(\frac{\partial \delta x(t,\tau)}{\partial t} + \frac{\partial \delta x(t,\tau)}{\partial \tau} + \mu(t,\tau)\delta x(t,\tau)$$
$$+ \delta u(t,\tau)\Big) d\tau dt - \int_0^\infty e^{-rt} \int_0^A \eta(t,\tau)\delta x(t,\tau) d\tau dt. \quad (16)$$

By integrating by parts and rearranging, we obtain:

$$\begin{aligned}
\delta\mathcal{L}(u,p) = & \int_0^\infty \int_0^A e^{-rt}\Big(c(t,\tau) - \lambda(t,\tau)\Big)\delta u(t,\tau) d\tau dt \\
& + \int_0^\infty e^{-rt}\Big(\lambda(t,0) - k(t)\Big)\delta p(t) dt \\
& + \int_0^\infty e^{-rt} \int_0^A \Big((r + \mu(t,\tau))\lambda(t,\tau) - \frac{\partial \lambda(t,\tau)}{\partial t} - \frac{\partial x(t,\tau)}{\partial \tau} \\
& - \beta(t,\tau)\Big)\delta x(t,\tau) d\tau dt + \int_0^A \lim_{t\to\infty} e^{-rt}\lambda(t,\tau)\delta x(t,\tau) d\tau \\
& + \int_0^\infty e^{-rt}\lambda(t,A)\delta x(t,A) dt.
\end{aligned} \quad (17)$$

Relation (17) leads to relations (10)-(13).
Under these relations we obtain :

$$\delta\mathcal{L}(u,p) = \int_0^\infty \int_0^A \frac{\partial I(u,p)}{\partial u}\delta u(t,\tau) d\tau dt + \int_0^\infty \int_0^A \frac{\partial I(u,p)}{\partial p}\delta p(t) d\tau dt. \quad (18)$$

Since (u^*, p^*) is a solution to problem (8), then we have:

$$\delta\mathcal{L}(u^*, p^*) = \mathcal{L}(u^* + \delta u, p^* + \delta p) - \mathcal{L}(u^*, p^*) \leq 0 \quad (19)$$

Let δu and δp be the admissible variations.
Let us first assume that $\delta u = 0$ and δp small, then from (19)

$$\mathcal{L}(u^*, p^* + \delta p) - \mathcal{L}(u^*, p^*) = \int_0^\infty \int_0^A \frac{\partial I(u^*, p^*)}{\partial p}\delta p d\tau dt \leq 0. \quad (20)$$

The previous inequality implies that the last three conditions of system (8) of the proposition are necessary for the optimality of (x^*, p^*).
Indeed, if one of the conditions is not valid, then there exists an admissible variation $\delta p_c(t)$ for all $t \in [0, \infty[$, which can be easily constructed such that $\mathcal{L}(u^*, p^* + \delta p_c) - \mathcal{L}(u^*, p^*) > 0$ which contradicts the fact that (x^*, p^*) is a solution.

Lemma 2.3.1 :
The optimal state variable $x^*(t,\tau)$ is equal to zero on a non-empty domain of the form $(t,\tau) \in \Delta \subset [0, \infty[\times[0, A[$, of strictly positive measure, i.e. the state constraint $x(t,\tau) \geq 0$ is active on Δ of non-zero measure [5].

To better understand the structure of the optimal trajectories, we will do a steady-state analysis of the problem.

2.4. Steady-state analysis

Steady-state analysis is an important step in the analytical study of population models. The basic principle is to assume that the optimization problem is autonomous, i.e., parameters do not explicitly depend on time t:

$$\mu(t,\tau)=\mu(\tau),\quad c(t,\tau)=c(\tau),\quad k(t)=k,$$

and search for possible stationary solutions

$$x(t,\tau)=x(\tau),\quad u(t,\tau)=u(\tau),\quad p(t)=p.$$

The stationary system of (2) is:

$$\begin{cases} \frac{dx(\tau)}{d\tau}=-\mu(\tau)x(\tau)-u(\tau) & \text{in } [0,A[\\ x(0)=p \\ x(\tau)\geq 0 \\ 0\leq p\leq p_{\max} \\ 0\leq u(\tau)\leq u_{\max}. \end{cases} \tag{21}$$

The first two equations of system (21) give us a Cauchy problem, so the solution exists and is unique. System (21) has a unique solution, which is given by :

$$x(\tau)=pe^{-\int_0^\tau \mu(s)ds}-\int_0^\tau u(s)e^{\int_\tau^s \mu(\sigma)d\sigma}ds. \tag{22}$$

The objective function is given by:

$$I=\int_0^{+\infty} e^{-rt}\left(\int_0^A c(\tau)u(\tau)d\tau-kp\right)dt.$$

The adjoint equation in the stationary case is given by:

$$\frac{d\lambda(\tau)}{d\tau}=(r+\mu(\tau))\lambda(\tau)-\eta(\tau),\quad \lambda(A)=0, \tag{23}$$

which is a first-order differential equation.
And has the following stationary solution:

$$\lambda(\tau)=\int_\tau^A e^{-\int_\tau^s (r+\mu(\sigma))d\sigma}\eta(s)ds. \tag{24}$$

We took the integration limits of A to τ because the variable $\lambda(\tau)$ vanishes at A.
That is to say $\lambda(A)=0$.

Bang-bang regime

Bang-bang regimes play an important role in the structure of solutions to linear optimization problems, such as problem (8). They reflect a situation in which a solution to the optimization problem mainly takes boundary values. Bang-bang regimes are also known in other scientific fields, e.g., in economics. The associated mathematical statements are called weak-form bang-bang theorems. The statement that a solution only takes extreme values is called the strong-form bang-bang theorem. It appears that a strong bang-bang principle is applicable for the steady-state harvest rate $u(\tau)$ in model (1)-(3) under realistic assumptions.

Theorem 2.4.1 :
If $u_{max}\gg 1$ and c is of class $\mathcal{C}^1$,

$$\frac{c'(\tau)}{c(\tau)}>r+\mu(\tau) \text{ at } 0\leq\tau\leq a^*\leq A, \tag{25}$$

and

$$\int_0^A c(\tau)e^{-\int_0^\tau \mu(s)ds}d\tau > k. \tag{26}$$

Then, the optimization problem (8) has the following steady state:

$$p^* = p_{max}, u^*(\tau) = \begin{cases} 0, & 0 < \tau \leq a^* \\ u_{max}, & a^* < \tau \leq a_e \\ 0, & a_e < \tau \leq A \end{cases} \tag{27}$$

$$x^*(\tau) = \begin{cases} > 0, & 0 < \tau \leq a_e \\ = 0 & a_e < \tau \leq A \end{cases} \tag{28}$$

where the endogenous harvest age a^*, $0 < a^* < A$, is determined from

$$a^* = arg \max_{0\leq\tau\leq A} \left[c(\tau)e^{-\int_0^\tau \mu(s)ds}\right], \tag{29}$$

and the endogenous age a_e, $a^* < a_e < A$, is found from $x(a_e) = 0$.

Proof :
The proof of this theorem is the same as that of [5].

3. Numerical resolution

Problems governed by partial differential equations are infinite-dimensional in nature. To obtain a numerical solution, we reduce the infinite-dimensional problem to a finite-dimensional one, to approximate a continuous problem to a discrete problem, which is more accessible to numerical tools. There are several methods that allow us to numerically approximate solutions to partial differential equations. In this article, we will apply the difference method to our problem. Throughout this section, we assume that:

- System (3) has a unique solution;
- We work on a finite horizon (T is fixed)
- The mortality rate is a bounded function and is constant with respect to time ($\mu(t,\tau) = \mu(\tau)$);
- The function x and all data are regular.

3.1. Principles of the finite difference method

The finite difference method for approximating the solution of a PDE is based on the approximation of derivatives by differences. The principle of this method is to replace the derivatives with approximate expressions calculated from the values of the function on a discrete grid. We will attempt to calculate an approximate solution at a finite number of points (t_j, τ_i) in the age-time domain $[0,T] \times [0,A]$. We will limit ourselves to the case where the mesh is regular: let N, M be two strictly positive integers. We set

$$t_j = j\Delta t \quad \forall j \in \{0,1,2,\cdots,M\}, \text{ where } \Delta t = \frac{T}{M},$$

$$\tau_i = i\Delta\tau \quad \forall i \in \{0,1,2,\cdots,N\}, \text{ where } \Delta\tau = \frac{A}{N}.$$

In particular, $t_0 = 0$, $t_M = T$, $\tau_0 = 0$ and $\tau_N = A$. The points (t_j, τ_i) are then intersection points of a regular age-time grid. The finite difference approximation then consists of seeking an approximation, denoted x_i^j, of $x(t_j, \tau_i)$.
The approximations of the first derivative $\frac{\partial x(t,\tau)}{\partial t}$ by finite differences are:

- centered finite differences
$$\frac{\partial x(t_j,\tau_i)}{\partial t} \approx \frac{x_i^{j+1} - x_i^{j-1}}{2\Delta t},$$
- right-offset finite differences
$$\frac{\partial x(t_j,\tau_i)}{\partial t} \approx \frac{x_i^{j+1} - x_i^{j}}{\Delta t},$$
- left-offset finite differences
$$\frac{\partial x(t_j,\tau_i)}{\partial t} \approx \frac{x_i^{j} - x_i^{j-1}}{\Delta t}$$
.

In the following we use only the last two approximations.

3.1.1. Application of the method to the model problem

We apply the finite difference method that we developed above to our model problem:

$$\begin{cases} \frac{\partial x(t,\tau)}{\partial t} + \frac{\partial x(t,\tau)}{\partial \tau} = -\mu(\tau)x(t,\tau) - u(t,\tau) & \text{in } [0,T]\times[0,A] \\ x(0,\tau) = x_0(\tau) & \text{in } [0,A] \\ x(t,0) = p(t) & \text{in } [0,T] \\ x(t,\tau) \geq 0 \\ 0 \leq p(t) \leq p_{max} \\ 0 \leq u(t,\tau) \leq u_{max}. \end{cases} \tag{30}$$

We denote by x_i^j the value of the approximate solution at point (t_j, τ_i) and by $x(t,\tau)$ the exact solution to the problem. The approximate values at the mesh points at $t = 0$ and $\tau = 0$ are given by :

$$x_0^j = p(j\Delta t) \qquad \forall j \in \{0, 1, 2, \cdots, M\}, \tag{31}$$

$$x_i^0 = x^0(i\Delta\tau) \qquad \forall i \in \{0, 1, 2, \cdots, N\}. \tag{32}$$

The approximations of the derivatives are given by :

$$\frac{\partial x(t_j,\tau_i)}{\partial t} \approx \frac{x_i^{j+1} - x_i^{j}}{\Delta t},$$
$$\frac{\partial x(t_j,\tau_i)}{\partial \tau} \approx \frac{x_i^{j} - x_{i-1}^{j}}{\Delta \tau}.$$

We use the implicit Euler scheme, the problem scheme is as follows:

$$\frac{x_i^{j+1} - x_i^{j}}{\Delta t} + \frac{x_i^{j+1} - x_{i-1}^{j+1}}{\Delta \tau} = -\mu(\tau_i)x_i^{j+1} - u(t_{j+1}, \tau_i) \tag{33}$$

who becomes,

$$\frac{x_i^{j+1} - x_i^{j}}{\Delta t} + \frac{x_i^{j+1} - x_{i-1}^{j+1}}{\Delta \tau} = -\mu_i x_i^{j+1} - u_i^{j+1}. \tag{34}$$

The discrete problem is to find:
x_i^j pour $\quad i = 1, \cdots, N; \quad j = 0, \cdots, M-1$ such as :

$$\begin{cases} (1+\alpha_i)x_i^{j+1} - \gamma x_{i-1}^{j+1} = x_i^j - \Delta t u_i^{j+1}, \quad \alpha_i = \frac{\Delta t}{\Delta \tau} + \Delta t \mu_i \text{ and } \gamma = \frac{\Delta t}{\Delta \tau} \\ x_0^{j+1} = p((j+1)\Delta t) \\ x_i^0 = x_0(i\Delta\tau). \end{cases} \tag{35}$$

Setting

$$X^j = (x_1^j, x_2^j, \cdots, x_N^j) \quad \forall j = 0, 1, 2, \cdots, M,$$

we obtain in matrix form:

$$LX^{j+1} = \gamma X^j + \Delta t C^{j+1} \tag{36}$$

with

$$L = \begin{pmatrix} 1+\alpha_1 & 0 & 0 & \cdots & 0 \\ -\gamma & 1+\alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\gamma & 1+\alpha_{N-1} & 0 \\ 0 & \cdots & 0 & -\gamma & 1+\alpha_N \end{pmatrix} \quad et \quad C^{j+1} = \begin{pmatrix} \frac{x_0^{j+1}}{\Delta\tau} - u_1^{j+1} \\ -u_2^{j+1} \\ -u_3^j \\ \vdots \\ -u_N^{j+1} \end{pmatrix},$$

L is a square matrix of order N and C^{j+1} is a vector of $\mathbb{R}^N$.
Scheme (36) is called implicit because, unlike the explicit scheme, its resolution requires a lot of computation at each time step to determine the approximate value. This shows that solving the implicit scheme is more expensive than the explicit scheme.
Nevertheless, this cost in computation time is largely offset by the scheme's greater stability.

4. Simulation

Numerical simulation is a powerful tool for solving and analyzing mathematical models when exact analytical solutions are difficult or impossible to obtain. It allows for approximating solutions and studying their behavior under various conditions.
The simulations below were performed in the MATLAB environment. We used field data from millet farming, obtained through a farmer survey, a highly prized commodity here in Niger:
$T = 1,\ A = 0.5, 1, 5, 6,\ \mu(\tau) = \frac{e^{-\tau}}{A},\ x_0 = e^{-\frac{\tau^2}{2}},\ u_{max} = 10,\ p_{max} = 20.$

Discussion

Figures 1, 2, 3 and 4 illustrate the evolution of controlled and uncontrolled densities in our model. We observe an increase in density at a relatively young age, due to the introduction of new individuals (Figure 1). We also observe a decrease in uncontrolled density with age, which continues to decline until reaching zero starting in Figure 3; this is also due to natural mortality. As for controlled density, we observe a rapid decline due to the combined effect of harvesting and natural mortality. In summary, we can say that from a certain age, even without harvesting, the population is stable, as shown in the last figures. It is therefore imperative to advise our farmers to sow their crops at the right time.

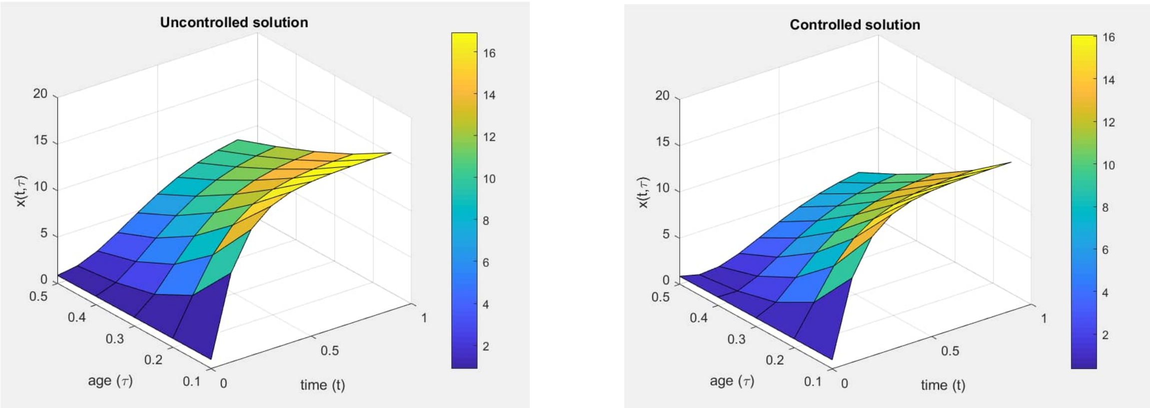

Figure 1: Evolution of solutions for $M = 10$ and $N = 5$

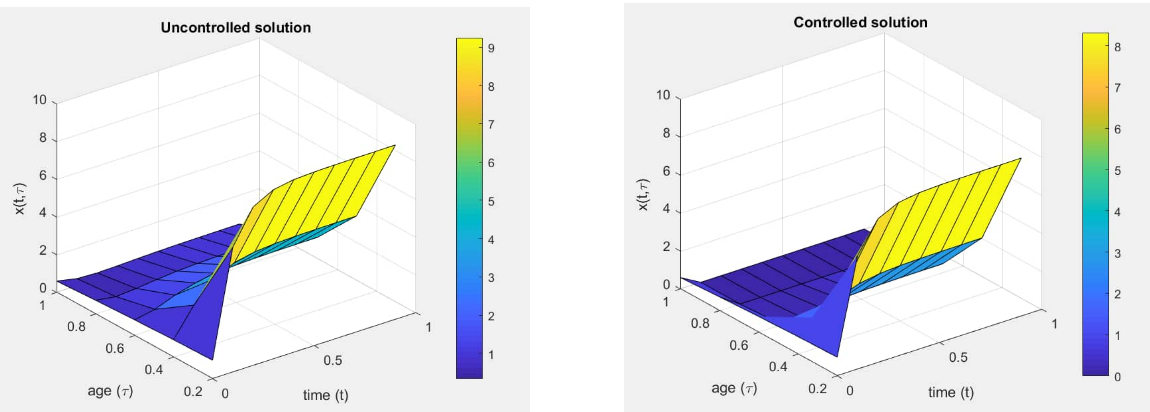

Figure 2: Evolution of solutions for $M = 10$ and $N = 5$

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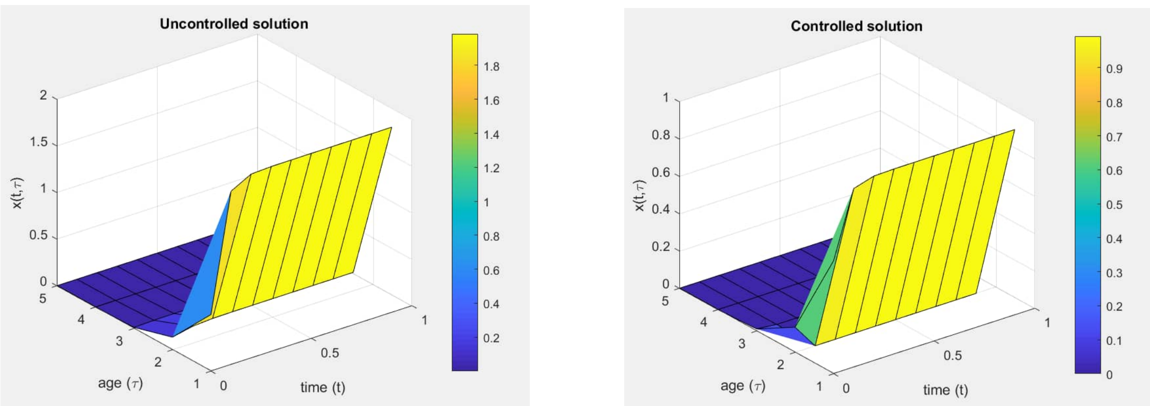

Figure 3: Evolution of solutions for $M = 10$ and $N = 5$

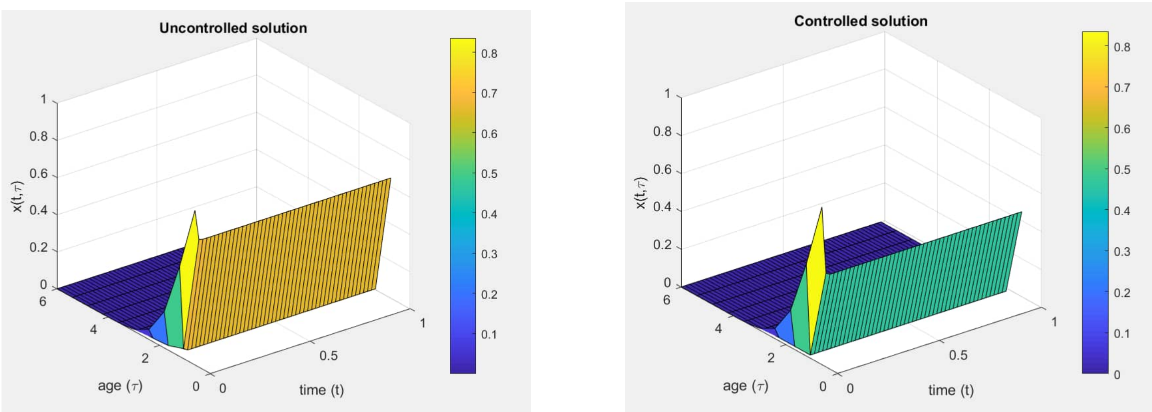

Figure 4: Evolution of solutions for $M = 50$ and $N = 10$

5. Conclusion

This study developed and analyzed an age-structured optimal harvest control model. Using a Lotka-Mckendrick partial differential equation approach, we demonstrated how profit can be maximized while ensuring population sustainability.

The numerical results highlighted the importance of parameters such as harvest and mortality rates. Using the finite difference method allowed us to numerically solve the problem, which can be applied to various real-world scenarios. This work contributes to a better understanding of sustainable natural resource management tools and paves the way for new practical applications in several fields.

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