Journal of Contemporary Applied Mathematics
V. 15, No 1, 2025, June
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam.151.8

Generalisation of algebraic point families of the Fermat curve quotient $\mathcal{C}_{1,2}(7)$

M.M.D.Diallo, C.M.Coly

Abstract. We explicitly determine the algebraic point families of a given degree over $\mathbb{Q}$ of the curve $\mathcal{C}_{1,2}(7)$ with affine equation:

$$y^7 = x(x-1)^2$$

This curve is a special case of the family of quotients of Fermat curves $\mathcal{C}_{r,s}(p)$ described in [11] of affine equation:

$$y^p = x^r(x-1)^s \quad \text{with} \quad 1 \leq r, \ s, \ r+s \leq p;$$

for $r = 1$, $s = 2$ and $p = 7$ such a curve was considered in [10]. This curve has been studied by O. Sall in [14], where the author gives a parametrisation of the cubic points. It should be noted, however, that the method used by O. Sall does not allow us to determine the set of points of degree greater than 3. We have therefore used a geometric method to extend this work and determine the quartic points [4]. In this note, we describe all the families of algebraic points of given degree, geometrically specifying the contact lines and the curve containing them, by applying the fundamental Abel-Jacobi theorem [1, 9], before using these results with a $\mathbb{Q}$-basis of the linear systems $\mathcal{L}(m\infty)$ and combining the contact order of the curve and specific points to obtain analytical expressions for these families of points.

Key Words and Phrases: Mordell-Weill group, Rational Points, Jacobian, Galois Conjugate, Linear Systems.

2010 Mathematics Subject Classifications: 14L40; 14H15; 14C20; 14H45.

1. Introduction

Algebraic geometry is a field of mathematics that historically has been interested in geometric objects (curves, surfaces, etc.) composed of points whose

coordinates verified equations that were not only sums and products.
Descartes' geometry [3] inaugurating the study of algebraic curves by the methods of analytic geometry, marks a great stage in the genesis of this discipline.
At the beginning of the twentieth century, algebraic geometry became a field in its own right.
This was initiated on the one hand by the work of David Hilbert [17, 15] and then was developed on the other hand by the Italian school at the end of the twentieth century.
Towards the end of the 1930s, André Weil introduced a formalism that made it possible to rigorously demonstrate their results (for a more extensive range of readers, see [16]).
In the aftermath of the 1930s in [12], the mathematicians Zariski, Brauer, Kolmogorov, Weil, and Chevalley developed in a more algebraic form the study of manifolds on any commutative field, essentially using ring theory.
In the 1950s, this branch had a great boom under the impetus of Pierre Samuel, Enri Cartan, Jean Pierre Serre and Alexandre Grothendieck.
Let p be a prime number. Let $\mathcal{F}_p$ be the Fermat curve of degree p defined by:

$$\mathcal{F}_p \ : \ X^p + Y^p = 1 \tag{1}$$

and $\mathcal{J}(p)$ its jacobian. Faddev (see [7]) prouve that when $p \leq 7$ the Mordell-Weil group of $\mathcal{J}(p)$ over $\mathbb{Q}$ is finite. Gross and Rohrlich show that, this group contains points of infinite order for $p > 7$ (cf. [10]).
In this document, we are interested in algebraic curves which are quotients of Fermat curves. These curves are described by affine equations of the type:

$$\mathcal{C}_{r,s}(p) \ : \ y^p = x^r(x-1)^s \qquad \text{with} \qquad 1 \leq r, \ s, \ r+s \leq p-1 \tag{2}$$

They have interested a large number of algebraic geometers, including Gross and Rohrlich who determined in 1978 the set of algebraic points of degree ≤ 2 for $p = 5, 7$ or 11.
In 2003 (see [13]), Sall extended the work of Gross and Rohrlich by determining the set of algebraic points of any degree ℓ for $p = 5, 7$ or 11.
The case $\mathcal{C}_{1,2}(7)$ was already studied in 1908 by Hurwitz (cf. [11]) who had described the rational points. In 2003 Sall extended (see [14]), by determining the set of algebraic points of degree at most 3 on the curve $\mathcal{C}_{1,2}(7)$.
In 2024, we extended the work of O. Sall by determining the quartic points on the same curve [4].
In this note, we explicitly determine the families of algebraic points of any degree on $\mathcal{C}_{1,2}(7)$, which makes the originality of the work.
The determination of algebraic points seems to be favorable when the Mordell-

Weil group of the Jacobian $\mathcal{J}(\mathbb{Q})$ is finite. It should be noted that the Riemann–Roch theorem plays an essential role in the calculations of the dimensions of vector spaces $\mathcal{L}(m\infty)$ *i.e* if $m \geq 2g - 1$ with m is a natural non-zero number; then $\dim(\mathcal{L}(m\infty)) = m - g + 1$ where g is the genus of the curve.
Let $\mathcal{C}$ be a projective algebraic curve defined over $\mathbb{Q}$. For any number field $\mathbb{K}$, we denote by $\mathcal{C}(\mathbb{K})$ the set of algebraic points on $\mathcal{C}$ with coordinates in $\mathbb{K}$ and by $\bigcup_{[\mathbb{K}:\mathbb{Q}]\leq d} \mathcal{C}(\mathbb{K})$ or $\mathcal{C}^d(\mathbb{K})$ the set of points on $\mathcal{C}$ with coordinates in $\mathbb{K}$ of degree at most d over $\mathbb{Q}$. The degree of a point P is the degree of its definition field over $\mathbb{Q}$, *i.e* $\deg(P) = [\mathbb{Q}(P : \mathbb{Q}]$.
We denote by $\mathcal{J}_{1,2}$ the jacobian of the curve $\mathcal{C}_{1,2}(7)$ with affine equation
$y^7 = x(x-1)^2$ and by $j_{1,2}(P)$ the class $[P - \infty]$ of $P - \infty$, i.e. j is the Jacobian folding:

$$\begin{array}{rccc} j_{1,2}: & \mathcal{C}_{1,2}(7) & \longrightarrow & \mathcal{J}_{1,2}(7)(\mathbb{Q}) \\ & P & \longmapsto & [P - \infty] \end{array} \tag{3}$$

where $\mathcal{J}_{1,2}(7)(\mathbb{Q})$ denotes the Mordell-Weil group of rational points of the Jacobian of $\mathcal{C}_{1,2}(7)$. This group is finite (see [6, 13]). This work consists in two parts, the first of which consists of developing certain essential results, called 'Auxiliary results', which enable us to demonstrate our result in the next section, called 'Main result'.

2. Auxiliary results

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and for a divisor D in $\mathcal{C}$, let $\mathcal{L}(D)$ as the $\bar{\mathbb{Q}}$ vector space of rational functions f defined over $\mathbb{Q}$ such that $f = 0$ or $div(f) \geq -D$; $l(D)$ denotes the $\bar{\mathbb{Q}}$ dimension of $\mathcal{L}(D)$.

Lemma 1. *According to [10], the Mordell-Weil group of the rational points of the Jacobian of* $\mathcal{C}_{1,2}(7)$ *is given by:*

$$\mathcal{J}_{1,2}(7)(\mathbb{Q}) \cong \mathbb{Z}/7\mathbb{Z}$$

Proof. This result is given in [14] and is explicitly proved in [10].

Let's P_0 , P_1, G_1, G_2 and ∞ are the points of the curve $\mathcal{C}_{1,2}(7)$ in the projective space $\mathbb{P}$ defined by: $P_0 = [0 : 0 : 1]$, $P_1 = [1 : 0 : 1]$, $G_1 = [\eta : \bar{\eta} : 1]$; $G_2 = [\bar{\eta} : \eta : 1]$ and $\infty = [1 : 0 : 0]$ where η is a primitive sixth root of unity and $\bar{\eta}$ is its complex conjugate.

Lemma 2. *We have the following rationnel divisors on the curve* $\mathcal{C}_{1,2}(7)$*:*

a) $div(x-\mu) = 7P_\mu - 7\infty \quad where \quad \mu \in \{0,1\}$,

b) $div(y) = \sum_{\mu=0}^{1}(\mu+1)P_\mu - 3\infty$.

Proof. A brief demonstration is given in [4], but a complete one is given in [14].

Corollary 1. *For our curve $\mathcal{C}_{1,2}(7)$ we have the following equations:*

a) $j_{1,2}(P_0) = -2j_{1,2}(P_1)$,

b) $7j_{1,2}(P_0) = 7j_{1,2}(P_1) = 0$.

Proof. They are direct consequences of Lemma 2 by associating the Jacobian plunge expression (3).

Lemma 3. *The Mordell-Weil group of rational points of the Jacobian of $\mathcal{C}_{1,2}(7)$ is generated by*

$$\mathcal{J}_{1,2}(7)(\mathbb{Q}) \cong \{mj_{1,2}(P_0)|\, m \in \{0,\ldots,6\}\} \cup \{mj_{1,2}(P_0) + y_0|\, m \in \{0,\ldots,6\}\},$$

with $y_0 = \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right]$.

Proof. See ([11], [14]).

Lemma 4.

1 *For our curve $\mathcal{C}_{1,2}(7)$, we obtain the following linear systems:*

- $\mathcal{L}(\infty) = \mathcal{L}(2\infty) = \langle 1 \rangle$,
- $\mathcal{L}(3\infty) = \mathcal{L}(4\infty) = \langle 1,\ y \rangle$,
- $\mathcal{L}(5\infty) = \left\langle 1,\ y,\ \frac{y^{\frac{19}{3}}}{(x-1)^2} \right\rangle$,
- $\mathcal{L}(6\infty) = \left\langle 1,\ y,\ \frac{y^{\frac{19}{3}}}{(x-1)^2},\ y^2 \right\rangle$,
- $\mathcal{L}(7\infty) = \left\langle 1,\ y,\ \frac{y^{\frac{19}{3}}}{(x-1)^2},\ y^2,\ x \right\rangle$,

- $\mathcal{L}(8\infty) = \left\langle 1,\ y,\ \frac{y^{\frac{19}{3}}}{(x-1)^2},\ y^2,\ x,\ \frac{y^{\frac{22}{3}}}{(x-1)^2} \right\rangle$,
- $\mathcal{L}(9\infty) = \left\langle 1,\ y,\ \frac{y^{\frac{19}{3}}}{(x-1)^2},\ y^2,\ x,\ \frac{y^{\frac{22}{3}}}{(x-1)^2},\ y^3 \right\rangle$,
- $\mathcal{L}(10\infty) = \left\langle 1,\ y,\ \frac{y^{\frac{19}{3}}}{(x-1)^2},\ y^2,\ x,\ \frac{y^{\frac{22}{3}}}{(x-1)^2},\ y^3,\ xy \right\rangle$.

2 *Generally, for all* $m \in \mathbb{N}$, *a* $\mathbb{Q}$*-base of the space* $\mathcal{L}(m\infty)$ *is given by:*

$$\mathcal{B}_m = \left\{ y^i \,\middle|\, i \leq \frac{m}{3} \right\} \bigcup \left\{ xy^j \,\middle|\, j \leq \frac{m-7}{3} \right\} \bigcup \left\{ \frac{y^{\frac{19}{3}+k}}{(x-1)^2} \,\middle|\, k \leq \frac{m-5}{3} \right\},$$

such that $(i,j,k) \in \mathbb{N}^3$

Proof. It is clear that $\mathcal{B}_m$ *is free. It remains to show that the cardinality of* $\mathcal{B}_m$ *is equal to the dimension of* $\mathcal{L}(m\infty)$. *For* $m \leq 2g-2 = 4$, Lemma 2 *and the Clifford's theorem [5] gives the results; which justifies the* **1**.
For the second point noted **2**, *we will consider* $m \geq 2g-1 = 5$, *we have according to Riemann-Roch theorem (cf.* [2, 8]*),* $\dim(\mathcal{L}(m\infty)) = m - g + 1$. *Let consider the following cases:*

⊙ 1^{st} ***Case***: *Consider* m *is even. Let* $m = 2(3h)$ *with* $h \in \mathbb{N}$; *we have:*

$*$ *for* i: $i \leq \frac{m}{3} \Leftrightarrow i \leq \frac{6h}{3} \Leftrightarrow i \leq 2h$,

$*$ *for* j: $j \leq \frac{m-7}{3} \Leftrightarrow j \leq \frac{6h-7}{3} \Rightarrow j < \frac{6h-6}{3} \Leftrightarrow j < 2h-2 \Rightarrow j \leq 2h-3$,

$*$ *for* k: $k \leq \frac{m-5}{3} \Leftrightarrow k \leq \frac{6h-5}{3} \Rightarrow k < \frac{6h-3}{3} \Leftrightarrow k < 2h-1 \Rightarrow k \leq 2h-2$.

Then, the expression of $\mathcal{B}_m$ *is given by:*

$$\mathcal{B}_m = \left\{1, y, \ldots, y^{2h}\right\} \bigcup \left\{x, xy, \ldots, xy^{2h-3}\right\} \bigcup \left\{ \frac{y^{\frac{19}{3}}}{(x-1)^2}, \ldots, \frac{y^{\frac{19}{3}+2h-2}}{(x-1)^2} \right\}.$$

This results in the dimension of $\mathcal{B}_m$ *being as follows:*

$$\#\mathcal{B}_m = (2h+1)+(2h-3+1)+(2h-2+1) = 6h-2 = m-2 = dim\mathcal{L}(m\infty).$$

⊙ 2^{nd} ***Case***: *Consider* m *is odd. Let* $m = 2(3h)+1$ *with* $h \in \mathbb{N}$; *we have:*

* *for i:* $i \leq \frac{m}{3} \Leftrightarrow i \leq \frac{6h+1}{3} \Rightarrow i < \frac{6h+3}{3} \Leftrightarrow i < 2h+1 \Rightarrow i \leq 2h,$
* *for j:* $j \leq \frac{m-7}{3} \Leftrightarrow j \leq \frac{6h-6}{3} \Leftrightarrow j \leq 2h-2,$
* *for k:* $k \leq \frac{m-5}{3} \Leftrightarrow k \leq \frac{6h-4}{3} \Rightarrow k < \frac{6h-3}{3} \Leftrightarrow k < 2h-1 \Rightarrow k \leq 2h-2.$

Then, the expression of $\mathcal{B}_m$ *is given by:*

$$\mathcal{B}_m = \left\{1, y, \dots, y^{2h}\right\} \bigcup \left\{x, xy, \dots, xy^{2h-2}\right\} \bigcup \left\{\frac{y^{\frac{19}{3}}}{(x-1)^2}, \dots, \frac{y^{\frac{19}{3}+2h-2}}{(x-1)^2}\right\}$$

This results in the dimension of $\mathcal{B}_m$ *being as follows:*

$$\#\mathcal{B}_m = (2h+1)+(2h-2+1)+(2h-2+1) = 6h-1 = m-2 = \dim \mathcal{L}(m\infty).$$

Proposition 1. *The geometric expression of the algebraic points is described by the intersections* Γ_n *and* $\Gamma_{n'}$ *with the curve* $\mathcal{C}_{1,2}(7)$ *over* $\mathbb{Q}$ *is given by:*

- $\Gamma_n \cdot \mathcal{C}_{1,2} = \sum_{\nu=1}^{d} R_\nu + (7n - d - 2m)\infty,$

- $\Gamma_{n'} \cdot \mathcal{C}_{1,2} = \sum_{\nu=1}^{d} R_\nu + 2mP_1 + \sum_{\kappa=1}^{2} G_\kappa + (7n' - d - 2m - 2)\infty,$

where Γ_n *and* $\Gamma_{n'}$ *are the curves defined by:*

$$\Gamma_n = \left\{[X:Y:1] \in \mathcal{C}_{1,2}(\mathbb{Q}) \middle| Z^n \xi\left(\frac{X}{Z}, \frac{Y}{Z}\right) = 0, \xi \in \mathcal{L}\left((d+2m)\infty\right)\right\},$$

$$\Gamma_{n'} = \left\{[X:Y:1] \in \mathcal{C}_{1,2}(\mathbb{Q}) \middle| Z^{n'} \zeta\left(\frac{X}{Z}, \frac{Y}{Z}\right) = 0, \zeta \in \mathcal{L}\left((d+2m+2)\infty\right)\right\}.$$

Proof. Let $R \in \mathcal{C}_{1,2}\ (7)(\mathbb{Q})$ with $[\mathbb{Q}(R):\mathbb{Q}] = d$ and R_ν for $\nu \in \{1, \dots, d\}$ the Galois conjugates of R. Let's work with $t = \left[\sum_{\nu=1}^{d} R_\nu - d\infty\right]$ which is a point of $\mathcal{J}_{1,2}(7)(\mathbb{Q})$. We will use two steps depending on whether t belongs to $\{mj_{1,2}(P_0) \mid 0 \leq m \leq 6\}$ or to $\{mj_{1,2}(P_0) + y_0 \mid 0 \leq m \leq 6\}$ with $y_0 = \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right]$.

First step: Let us first consider t in the set $\{mj_{1,2}(P_0) \mid 0 \leq m \leq 6\}$. This implies that $t = mj_{1,2}(P_0)$ for $0 \leq m \leq 6$ and furthermore $t = \left[\sum_{\nu=1}^{d} R_\nu - d\infty\right]$. We then obtain the following equation:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = mj_{1,2}(P_0), \tag{4}$$

from Corollary 1, the equation (4) becomes:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = -2mj_{1,2}(P_1), \tag{5}$$

by applying the expression for the Jacobian fold noted in (3) to equation (5), we obtain the following equality:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = -\left[2mP_1 - 2m\infty\right], \tag{6}$$

from expression (6), we can then deduce the following equation:

$$\left[\sum_{\nu=1}^{d} R_\nu + 2mP_1 - (d+2m)\infty\right] = 0. \tag{7}$$

According to the Abel-Jacobi theorem [1, 9], from equation (7), there exists a rational function $\xi(x, y)$ defined on $\mathbb{Q}$ such that the principal divisor is described as follows:

$$div(\xi) = \sum_{\nu=1}^{d} R_\nu + 2mP_1 - (d+2m)\,\infty, \tag{8}$$

for $0 \leq m \leq 6$; then $\xi \in \mathcal{L}\left((d+2m)\infty\right)$. The function ξ is a polynomial of $n \leq \lfloor \frac{d+19}{3} \rfloor$ where $\lfloor \frac{d+19}{3} \rfloor$ denotes the integer part of $\frac{d+19}{3}$. There is then a curve Γ_n defined over $\mathbb{Q}$ of equation $Z^n \xi(\frac{X}{Z}, \frac{Y}{Z}) = 0$ such that

$$\Gamma_n \cdot \mathcal{C}_{1,2} = \sum_{\nu=1}^{d} R_\nu - (d+2m)\infty + 7n\infty, \tag{9}$$

we conclude from the expression (9) that $\Gamma_n \cdot \mathcal{C}_{1,2}$ is expressed as follows:

$$\Gamma_n \cdot \mathcal{C}_{1,2} = \sum_{\nu=1}^{d} R_\nu + (7n - d - 2m)\infty.$$

Second step: In this second part, we take t in the set $\{mj_{1,2}(P_0) + y_0 \mid 0 \le m \le 6\}$. This implies that $t = \{mj_{1,2}(P_0) + y_0 \mid 0 \le m \le 6\}$. This implies that $t = \{mj_{1,2}(P_0) + y_0 \mid 0 \le m \le 6\}$, furthermore $t = \left[\sum_{\nu=1}^{d} R_\nu - d\infty\right]$ and $y_0 = \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right]$. We then obtain the following equation:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = mj_{1,2}(P_0) + \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right], \qquad (10)$$

from Corollary 1, the equation (10) becomes:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = mj_{1,2}(P_0) + \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right], \qquad (11)$$

by reapplying the expression for the Jacobian fold noted in (3) to equation (11), we obtain the following equality:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = -2mj_{1,2}(P_1) + \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right], \qquad (12)$$

expression (12), we can then deduce the following equation:

$$\left[\sum_{\nu=1}^{d} R_\nu - d\infty\right] = -[2mP_1 - 2m\infty] + \left[-\sum_{\kappa=1}^{2} G_\kappa + 2\infty\right], \qquad (13)$$

then, we have the following equation:

$$\left[\sum_{\nu=1}^{d} R_\nu + 2mP_1 + \sum_{\kappa=1}^{2} G_\kappa - (d + 2m + 2)\infty\right] = 0. \qquad (14)$$

Similarly, by application of the Abel-Jacobi theorem [1, 9], from equation (14), there exists a rational function $\zeta(x, y)$ defined on $\mathbb{Q}$ such that the principal divisor is defined as follows:

$$div(\zeta) = \sum_{\nu=1}^{d} R_\nu + 2mP_1 + \sum_{\kappa=1}^{2} G_\kappa - (d + 2m + 2)\infty, \qquad (15)$$

with $0 \leq m \leq 6$; then $\zeta \in \mathcal{L}((d+2m+2)\infty)$. The function ζ is a polynomial of degree $n' \leq \lfloor \frac{d+21}{3} \rfloor$ where $\lfloor \frac{d+21}{3} \rfloor$ denotes the integer part of $\frac{d+21}{3}$. There is then a curve $\Gamma_{n'}$ defined over $\mathbb{Q}$ of equation $Z^{n'}\zeta(\frac{X}{Z}, \frac{Y}{Z}) = 0$ such that

$$\Gamma_{n'} \cdot \mathcal{C}_{1,2} = \sum_{\nu=1}^{d} R_\nu + 2mP_1 + \sum_{\kappa=1}^{2} G_\kappa - (d+2m+2)\infty + 7n'\infty, \qquad (16)$$

we conclude from the expression (16) that $\Gamma_{n'} \cdot \mathcal{C}_{1,2}$ is expressed as follows:

$$\Gamma_{n'} \cdot \mathcal{C}_{1,2} = \sum_{\nu=1}^{d} R_\nu + 2mP_1 + \sum_{\kappa=1}^{2} G_\kappa + (7n' - d - 2m - 2)\infty.$$

3. Main result

Our main result is described by the following theorem:

Theorem 1. *The families of algebraic points of given degree on the quotient curve $\mathcal{C}_{1,2}(7)$ of affine equation $y^7 = x(x-1)^2$ are given as follows $\mathcal{C}^d_{1,2}(7)(\mathbb{Q}) = \mathcal{M} \bigcup \mathcal{D}$ such that*

$$\mathcal{M} = \left\{ \left(-\frac{\sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i}{\zeta(y)}, y \right) \middle| \begin{array}{c} a_{\frac{7n-d-2m}{3}} \neq 0 \text{ if } n \text{ and } d \text{ have the same} \\ \text{parity, } c_{\frac{7n-d-2m-5}{3}} \neq 0 \text{ otherwise, } a_0 \\ \text{and } b_0 \text{ cannot both be zero and } x \\ \text{is a root of the equation } (\mathcal{E}_1) \end{array} \right\}$$

and

$$\mathcal{D} = \left\{ \left(-\frac{\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t)}{\vartheta(t) + \beta(t)}, t \right) \middle| \begin{array}{c} a_{\frac{7n'-d-2m-2}{3}} \neq 0 \text{ if } n' \text{ and } d \text{ have} \\ \text{the same, } c_{\frac{7n'-d-2m-7}{3}} \neq 0 \text{ otherwise} \\ \text{and } x \text{ isa root of the equation } (\mathcal{E}_2) \end{array} \right\}$$

where

$$(\mathcal{E}_1)\ :\ y^7\left(\zeta(y)\right)^3=\sum_{i=0}^{\frac{7n-d-2m}{3}}a_iy^i\left(\sum_{i=0}^{\frac{7n-d-2m}{3}}a_iy^i+\zeta(y)\right)^2\qquad and$$

$$(\mathcal{E}_2)\ :\ y^7\left(\vartheta(t)+\beta(t)\right)^3=\left(\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}}a_it^i+\alpha(t)\right)\left(\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}}a_it^i+\alpha(t)+\vartheta(t)\right)^2,$$

with $s=(x-\mathcal{R}e(\eta))$, $t^{\varsigma}=y^{\varsigma}-\cos\left(\varsigma\arg(\eta)\right)$ *for* $\varsigma\in\{i,j,k\}$ *and* $\zeta(y)$, $\alpha(t)$, $\beta(t)$ *and* $\vartheta(t)$ *are polynomial functions defined by:*

$$\begin{cases}\zeta(y) &= \displaystyle\sum_{j=0}^{\frac{7n-d-2m-7}{3}}b_jy^j+\sum_{k=0}^{\frac{7n-d-2m-5}{3}}c_ky^{-\frac{2}{3}+k},\\ \alpha(t) &= \displaystyle-\sum_{\ell=1}^{2m-1}\left(b_\ell+\left(\frac{3\ell-2}{3\ell}\right)c_\ell\right)t^\ell,\\ \beta(t) &= \displaystyle-\left(\sum_{\ell=1}^{2m-1}\left(a_\ell+\left(\frac{3\ell-2}{3\ell}\right)c_\ell\right)t^\ell+\sum_{\ell=1}^{2m-1}\left(\frac{3\ell-2}{3\ell}\right)\left(a_\ell+b_\ell\right)t^{\ell-\frac{2}{3}}\right),\\ \vartheta(t) &= \displaystyle\sum_{j=2m}^{\frac{7n'-d-2m-9}{3}}b_jt^j+\sum_{k=2m}^{\frac{7n'-d-2m-7}{3}}c_kt^{k-\frac{2}{3}}.\end{cases}$$

Proof. First, note that if R is an algebraic point of degree d, then it belongs to the intersection family of the curve written in Proposition 1. Two cases are possible:

1^st^ Consider first that R is an element of $\Gamma_n\cdot\mathcal{C}_{1,2}$. Then there is a rational function $\chi(x,y)$ defined on $\mathbb{Q}$ such that $\chi(x,y)\in\mathcal{L}\left((7n-d-2m)\infty\right)$. From Lemma 4, we derive the expression $\chi(x,y)$ as follows:

$$\chi(x,y)=\sum_{i=0}^{\frac{7n-d-2m}{3}}a_iy^i+\sum_{j=0}^{\frac{7n-d-2m-7}{3}}b_jxy^j+\sum_{k=0}^{\frac{7n-d-2m-5}{3}}c_k\frac{y^{\frac{19}{3}+k}}{(x-1)^2},\qquad(17)$$

with a_i, b_j and c_k scalars belonging to $\mathbb{Q}$ such that a_0 and b_0 not simultaneously equal to zero (otherwise one of the R_ν's would have to be equal to P_μ with $\mu\in\{0,1\}$, which would be an absurd thing to do) $a_{\frac{7n-d-2m}{3}}\neq 0$ if n and d have the same parity (otherwise one of the R_ν's would have to be equal to ∞, which would be an absurd thing to do), $c_{\frac{7n-d-2m-5}{3}}\neq 0$ if n and d have different parities (otherwise one of the R_ν's would have to be equal to ∞, which would be an absurd thing to do). Since the points

R_ν are simple zeros of $\chi(x,y)$, we deduce that $\chi(x,y)=0$, which simplifies expression (17) to:

$$\sum_{j=0}^{\frac{7n-d-2m-7}{3}} b_j x y^j + \sum_{k=0}^{\frac{7n-d-2m-5}{3}} c_k \frac{y^{\frac{19}{3}+k}}{(x-1)^2} = -\sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i. \qquad (18)$$

Note that $\dfrac{y^{\frac{19}{3}+k}}{(x-1)^2} = \dfrac{y^7}{(x-1)^2} \times y^{-\frac{2}{3}+k} = xy^{-\frac{2}{3}+k}$. So equation (18) becomes:

$$\sum_{j=0}^{\frac{7n-d-2m-7}{3}} b_j x y^j + \sum_{k=0}^{\frac{7n-d-2m-5}{3}} c_k x y^{-\frac{2}{3}+k} = -\sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i. \qquad (19)$$

From equation (19), we can derive the expression for x as a function of y as follows:

$$x = -\frac{\displaystyle\sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i}{\displaystyle\sum_{j=0}^{\frac{7n-d-2m-7}{3}} b_j y^j + \sum_{k=0}^{\frac{7n-d-2m-5}{3}} c_k y^{-\frac{2}{3}+k}}. \qquad (20)$$

By replacing the value of x in expression (20) of the equation of curve $\mathcal{C}_{1,2}(7)$, we obtain:

$$(\mathcal{E}_1)\ :\ y^7\left(\zeta(y)\right)^3 = \sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i \left(\sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i + \zeta(y) \right)^2,$$

with $\zeta(y)$ the polynomial in y described by:

$$\zeta(y) = \sum_{j=0}^{\frac{7n-d-2m-7}{3}} b_j y^j + \sum_{k=0}^{\frac{7n-d-2m-5}{3}} c_k y^{-\frac{2}{3}+k}.$$

This gives a a first family of points of degree at most d given by:

$$\mathcal{M} = \left\{ \left(-\frac{\displaystyle\sum_{i=0}^{\frac{7n-d-2m}{3}} a_i y^i}{\zeta(y)}, y \right) \middle| \begin{array}{c} a_{\frac{7n-d-2m}{3}} \neq 0 \text{ if } n \text{ and } d \text{ have the same} \\ \text{parity, } c_{\frac{7n-d-2m-5}{3}} \neq 0 \text{ otherwise, } a_0 \\ \text{and } b_0 \text{ cannot both be zero and } x \\ \text{is a root of the equation } (\mathcal{E}_1) \end{array} \right\}$$

2nd Finally, let R be an element of $\Gamma_{n'} \cdot \mathcal{C}_{1,2}$.

Then there is a rational function $\varphi(x,y)$ defined on $\mathbb{Q}$ such that $\varphi(x,y) \in \mathcal{L}\left((7n'-d-2m-2)\infty\right)$ with $ord_{P_1}\varphi = 2m$ and $ord_{G_\kappa}\varphi = 1$ for any integer $\kappa \in \{1,2\}$. From Lemma 4, we derive the expression $\varphi(x,y)$ as follows:

$$\varphi(x,y) = \sum_{i=0}^{\frac{7n'-d-2m-2}{3}} a_i y^i + \sum_{j=0}^{\frac{7n'-d-2m-9}{3}} b_j x y^j + \sum_{k=0}^{\frac{7n'-d-2m-7}{3}} c_k \frac{y^{\frac{19}{3}+k}}{(x-1)^2}. \quad (21)$$

Furthermore, given that $\dfrac{y^{\frac{19}{3}+k}}{(x-1)^2} = xy^{-\frac{2}{3}+k}$, we have deduced from expression (21) a new writing of $\varphi(x,y)$ by:

$$\varphi(x,y) = \sum_{i=0}^{\frac{7n'-d-2m-2}{3}} a_i y^i + \sum_{j=0}^{\frac{7n'-d-2m-9}{3}} b_j x y^j + \sum_{k=0}^{\frac{7n'-d-2m-7}{3}} c_k x y^{k-\frac{2}{3}}. \quad (22)$$

Since on the one hand $ord_{G_\kappa}\varphi = 1$ for $\kappa \in \{1,2\}$, noting s and t the changes of variable associated with x followed by y respectively gives $s = (x - \mathcal{R}e(\eta))$ and $t^{\varsigma} = y^{\varsigma} - \cos(\varsigma \arg(\eta))$ with $\varsigma \in \{i,j,k\}$, then the expression (22) of $\varphi(s,t)$ will be written as follows:

$$\varphi(s,t) = \sum_{i=1}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \sum_{j=1}^{\frac{7n'-d-2m-9}{3}} b_j s t^j + \sum_{k=1}^{\frac{7n'-d-2m-7}{3}} c_k s t^{k-\frac{2}{3}}, \quad (23)$$

and on the other hand, since $ord_{P_1}\varphi = 2m$ imples that $a_\ell + b_\ell + \left(\dfrac{3\ell-2}{3\ell}\right) c_\ell = 0$ for $1 \le \ell \le 2m-1$ then the expression (23) of $\varphi(s,t)$ will be written as follows:

$$\varphi(s,t) = \sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t) + s\left(\sum_{j=2m}^{\frac{7n'-d-2m-9}{3}} b_j t^j + \sum_{k=2m}^{\frac{7n'-d-2m-7}{3}} c_k t^{k-\frac{2}{3}} + \beta(t)\right), \quad (24)$$

where $\alpha(t)$ and $\beta(t)$ are polynomial functions defined as follows

$$\begin{aligned} \alpha(t) &= -\sum_{\ell=1}^{2m-1} \left(b_\ell + \left(\frac{3\ell-2}{3\ell}\right) c_\ell\right) t^\ell \qquad \text{and} \\ \beta(t) &= -\left(\sum_{\ell=1}^{2m-1} \left(a_\ell + \left(\frac{3\ell-2}{3\ell}\right) c_\ell\right) t^\ell + \sum_{\ell=1}^{2m-1} \left(\frac{3\ell-2}{3\ell}\right) (a_\ell + b_\ell)\, t^{\ell-\frac{2}{3}}\right). \end{aligned}$$

such that a_i, b_j and c_k scalars belonging to $\mathbb{Q}$ such that $a_{\frac{7n'-d-2m-2}{3}} \neq 0$ if n' and d have the same parity (otherwise one of the R_ν's would have to be equal to ∞, which would be an absurd thing to do), $c_{\frac{7n'-d-2m-7}{3}} \neq 0$ if n' and d have different parities (otherwise one of the R_ν's would have to be equal to ∞, which would be an absurd thing to do). Furthermore, since R_ν is a simple zero of $\varphi(s,t)$, we deduce that $\varphi(s,t) = 0$, which simplifies expression (24) to:

$$s\left(\sum_{j=2m}^{\frac{7n'-d-2m-9}{3}} b_j t^j + \sum_{k=2m}^{\frac{7n'-d-2m-7}{3}} c_k t^{k-\frac{2}{3}} + \beta(t)\right) = -\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t) \qquad (25)$$

From equation (25), it follows that x is expressed as a function of y by:

$$s = -\frac{\displaystyle\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t)}{\displaystyle\sum_{j=2m}^{\frac{7n'-d-2m-9}{3}} b_j t^j + \sum_{k=2m}^{\frac{7n'-d-2m-7}{3}} c_k t^{k-\frac{2}{3}} + \beta(t)}. \qquad (26)$$

By replacing the value of x in the expression (26) of the equation of the curve $\mathcal{C}_{1,2}(7)$, we obtain:

$$(\mathcal{E}_2)\ :\ y^7\left(\vartheta(t) + \beta(t)\right)^3 = \left(\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t)\right)\left(\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t) + \vartheta(t)\right)^2,$$

with $\vartheta(t)$ the polynomial in t described by:

$$\vartheta(t) = \sum_{j=2m}^{\frac{7n'-d-2m-9}{3}} b_j t^j + \sum_{k=2m}^{\frac{7n'-d-2m-7}{3}} c_k t^{k-\frac{2}{3}}.$$

This gives a second family of points of degree at most d given by:

$$\mathcal{D} = \left\{\left(-\frac{\displaystyle\sum_{i=2m}^{\frac{7n'-d-2m-2}{3}} a_i t^i + \alpha(t)}{\vartheta(t) + \beta(t)}, t\right) \middle| \begin{array}{c} a_{\frac{7n'-d-2m-2}{3}} \neq 0 \text{ if } n' \text{ and } d \text{ have} \\ \text{the same, } c_{\frac{7n'-d-2m-7}{3}} \neq 0 \text{ otherwise} \\ \text{and } x \text{ isa root of the equation } (\mathcal{E}_2) \end{array}\right\}$$

Acknowledgement. Special thanks are due to the editor and his team at the Journal of Contemporary Applied Mathematics (JCAM).

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, The Basic Results of the Brill-Noether Theory, Geometry of Algebraic Curves, v.133, No.3, 1985, pp. 203–224.

[2] A. Borel and J.P. Serre, Le théoréme de Riemann-Roch, Bultin de la Société mathématiques de france, v.86, 1958, pp. 97–136.

[3] H. Bos, The structure of Descartes's Géométrie, Revue d'histoire des sciences, v.51, No.2, 1998, pp. 291–318.

[4] C.M. Coly, M.M.D. Diallo and M. Fall, Quartic Points on $\mathcal{C}_{1,2}(7)$, Nonlinear Analysis, Geometry and Applications: Proceedings of the Third NLAGA-BIRS Symposium, AIMS-Mbour, Senegal, August 2023, pp. 353–358.

[5] M. Coppens and G. Martens, Secant spaces and Clifford's theorem, Compositio Mathematica, v.78, No.2, 1991, pp. 193–212.

[6] D. Faddeev, *On the divisor class groups of some algebraic curves*, Dokl. Akad. Nauk SSSR 136 (1961) 296-298. English translation: Soviet Math. Dokl, v.2, No.1, 1961, pp. 67–69.

[7] D. K. Faddeev, Invariants of divisor classes for the curves $x^k(1-x) = y^\ell$ in an ℓ-adic cyclotomic field, Trudy Matematicheskogo Instituta imeni VA Steklova, v.64, 1961, pp. 284–293.

[8] G. Faltings, Lectures on the arithmetic Riemann-Roch theorem, Princeton University Press, v.127, 1992.

[9] P. A. Griffiths, Introduction to algebraic curves, Translations of mathematical monographs, American Mathematical Society, Providence, RI, v.76, 1989.

[10] B. H. Gross and D. E. Rohrlich, Some results on the Mordell-Weil group of the Jacobian of the Fermat curve, Inventiones mathematicae, v.44, 1978, pp. 201–224.

[11] A. Hurwitz, Über die diophantische Cleichung $x^3y + y^3z + z^3x = 0$, Mathematische Annalen, v.65, 1908, pp. 428–430.

[12] I. Kleiner, Biographies of Selected Mathematicians, A History of Abstract Algebra, 2007, pp. 113–163.

[13] O. Sall, Points algébriques sur certains quotients de courbes de Fermat, Comptes Rendus. Mathématiques, v.336, No.2, 2003, pp. 117–120.

[14] O. Sall, Points cubiques sur la quartique de Klein, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, v.333, No.10, 2001, pp. 931–934.

[15] N. Schappacher and H. David, Report on algebraic number fields ('Zahlbericht'), (1897), 1640-1940,, Landmark Writings in Western Mathematics, 2005, pp. 700–709.

[16] Weintraub, E., Mirowski, P. The pure and the applied: Bourbakism comes to mathematical economics, Science in Context, v.7,No.2, 1994, pp. 245–272.

[17] H. Weyl, *David Hilbert* , The Royal Society London, 1944, pp. 1862-1943.

Mohamadou M. D. Diallo
Department of Mathematics, Assane Seck University of Ziguinchor, 27000, Ziguinchor, Senegal
E-mail: m.diallo1836@zig.univ.sn

Cherif M. Coly
Department of Mathematics, Assane Seck University of Ziguinchor, 27000, Ziguinchor, Senegal
E-mail: c.coly1309@zig.univ.sn

Received 15 November 2024
Accepted 11 March 2025