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On the structure of tensor fields given on manifolds

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Abstract. In the present work, we consider the metric questions of diffeomorphic mappings of the manifolds. We find such bases on tangential spaces of manifolds corresponding to given mappings, which allow investigate their metric properties, independent of connection coefficients, by using of strictly analysis' methods. For simplicity of our considerations, we suffice with consideration of manifolds defined by finite number of maps. When our consideration is restricted with neighborhoods of some points, we shall suffice with one map defining the manifold.

Key Words and Phrases: Tensor fields, Jacobian matrix, characteristic equation, singular numbers, spectral map

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1. List of notations

We shall throughout the article use following notations.

$\mathbf{R}$-the set of real numbers;

$\mathbf{f}$-vector-function of several real variables;

$\mathbf{f}'$-derivative for the vector-function, that is, linear operator;

$(x,\, y)$-scalar product of vectors;

$D_{\bar{\xi}}$-derivative in the direction of the vector $\bar{\xi}$;

${}^tA$-transposing of the matrix A;

$C^{(k)}(U)$- set of smooth functions up to the order k in the domain U;

$\mathbf{T}_{\varphi}$-tangential space to the manifold defined by the map φ;

$P * Q$ -matrix set up by taking tensor product of columns with equal numbers for matrices P and Q of equal number of columns.

f-scalar function, that is function with real values;

$\nabla \mathbf{f}$-gradient vector of the scalar function f;

F_1 - matrix of the linear operator $\mathbf{f}'$.

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2. Introduction

The notion of vector fields arises in phisics and mechanics. For example, at every point of the space around the point charge it arises the electric field. At every point of the field another charged point, with positive unite charge, feels a force which is denoted as a vector. One calls this vector as an intensity vector, and set of all such vectors is called to be a vector field of intensity, or intensity field. In difference with an algebraic notion of the field, this notion is not demanding the set of such vectors be supplied and be closed with respect to operations over the vectors. Every vector of the field is a value for some vector function at the taken point, the image of which not obliged to contain results of operations under different vectors. So, defining of vector field is equivalent to giving of a vector function ([9]).

In many questions of mechanics, analysis and differential geometry the notion of tensor field is used. But due to existence of great number of applications, tensor fields are studing in such sections of mathematical analysis as the theory of differential equations with partial derivatives, variation calculus and others. In the paper [1] one studies the tensor fields generated by one function of several variables. In the present paper, we shall investigate a "structure" of tensor fields given on differentiable manifolds. Under the notion of the structure, we mean that properties of tensor fields which connected with differentiable maps and does not depend on the system of coordinates (that is, on parameterization). For example, in questions related to quadratic forms reducing of them to the sum of squares can be performed independent of change of coordintes. In difference with the methods of differential geometry, we shall not use upper and lower indexing, using isomorphism of the dual space to the initial one.

Note that in differential geometry, when differentiable mappings of manifolds are studing, arising coefficients of connection (or Cristoffel symbols) hinders obtaining of tensor properties of mappings. To overcome such difficulties, the special notions, as covariant derivative or parallel shifting, are used. In the present work, we try to find such bases on tangential spaces of manifolds corresponding to given mappings, which allow investigate their metric properties, independent of connection coefficients. In other words, we bypass an influence of connection coefficients, that is, overcome the influence of inner geometry of a manifold by using of strictly analysis' methods.

When manifold is given by the system of functions (that is, in parametric view), then to every its point is possible put in correspondence some linear operator (matrix). Considering this matrix in some neighborhood of a taken point as a functional matrix, we shall define a new linear operator given in the tensor products of tangential spaces. By using these operators, differentials of high order are possible represente as multilinear forms. In several metric questions, very complicated problem on investigation of such forms arises. What is why, we realize such an explanation in which is possible giving a procedure for clarifying structure of tensor fields. Sheme of such reduction was proposed in [1] for one differentable function in n-dimensional case (without proof). In this paper we consider similar questions in manifolds.

3. Metric questions on manifolds

Here we consider the metric questions of diffeomorphic mappings of the manifold M. For simplicity of our considerations, we suffice with consideration of manifolds defined by finite number of maps. When our consideration is restricted with neighborhoods of some points, we shall suffice with one map defining the manifold.

Let we are given with m-dimensional real manifold beeing given in $n-$ dimensional space. Suppose, that the manifold M has a smoothness of order k and defined by following parametric representation:

$$\varphi : x_i = \varphi_i(u_1, ..., u_m);\ i = 1, ..., n. \tag{1}$$

Suppose, that these functions are defined in the domain $U \subset R^m$, $m \leq n$.

Denote by N an image of the manifold M in some diffeomorphism f. We suppose that the mapping f has a reqired smoothness. Then the composition $g = f \circ \varphi$ will be a diffeomrphic map acting from R^m to the m-dimensional manifold $N \subset R^n$: $g \in C^{(k)}(\mathbf{R}^m)$. In the tangential space of given manifold, placed at arbitrary point $\bar{x} = \varphi(\bar{u})$ over the point $\bar{u} \in U$, take vectors tangential to the coordinate functions, that is, vectors $({}^t\nabla \mathbf{g_i} = (g'_{i1}, ..., g'_{im}))_{\bar{u} \in U}$, where $g'_{ij} = \partial g_i / \partial u_j$ (the symbol ^{t} in the left hand side over ∇ denotes transposing). It is clear that the matrix

$$G_1(\bar{u}) = \left(g'_{ij}\right)_{1 \leq i,j \leq m} = \begin{pmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_2}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_1} \\ \frac{\partial g_1}{\partial u_2} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_m}{\partial u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial u_m} & \frac{\partial g_2}{\partial u_m} & \cdots & \frac{\partial g_m}{\partial u_m} \end{pmatrix},$$

defined as a transposed Jacobian matrix of the system of coordinate functions of the vector $g \in C^{(k)}(\mathbf{R}^m)$, defines a linear operator in this tangential space. Due to rule of differentiation for composite function, the matrix $G_1(\bar{u})$ is possible to represent as a product of derivative matrices ([3]) of maps f and φ. Then every vector in $U \subset R^m$ transforms, by the matrix $G(\bar{u})$, to some vector of tangential space of the manifold N. Denoting by $\mathbf{T}_\varphi$ the tangential space of the manifold given by φ (that is, the space $\mathbf{R}^m$), we note that the matrix ${}^tG_1(\bar{u})$ defines a linear map ${}^tG_1 : \mathbf{T}_\varphi \to \mathbf{T}_g$ over the point $\bar{u}$. So, differentiation produces a linear map of tangential spaces (which is well known from analysis and geometry). Below we shall omit such comments, considering diffeomorphism g.

The matrix $G_1(\bar{u})$, at the same time, defines a field (vector function) connected to the diffeomorphism g : $\mathbf{g}_1 = ({}^t\nabla\mathbf{g_1}, ..., {}^t\nabla\mathbf{g_m})$, if to consider it as an element of the space $\mathbf{R}^{m^2}$. Consider derivative $\mathbf{g}_1'$ of this vector function and take its matrix $G_2(\bar{u})$, that is, Jacobian matrix for the system of gradients. We can consider this linear operator as an operator acting from tensor products of tangential spaces$\mathbf{T}_\varphi \otimes \mathbf{T}_\varphi$. So, we can consider the field $\mathbf{g_1} = \mathbf{g}_1(\bar{u})$ as a tensor field. The structure of this field well defined by singular value decomposition of the matrix $G_2(\bar{u})$, which allows every its entry express by singular numbers (that is, by m number of parameters).

Despite that the dimensions of tangential spaces are increasing, actually these fields are defined by m number of parameters. Denote singular numbers of the matrix $G_2(\bar{u})$ as $\lambda_1(\bar{u}), ..., \lambda_m(\bar{u})$. These functions are roots of characteristic equation which is some algebraic equation with functional coefficients. As it is best known (see [3, 4]), this equation uniquely defines singular numbers as a system of implicit functions in some neighborhood of every point, at which the discriminant of the characteristic equation is distinct from zero, with non-singular Jacobian matrix. These functions, in the specifed neighborhood, are differentiable. This is a consequence of the Lemma 1 below ([4]).

Lemma 1. *Let we are given with an algebraic equation*

$$z^n - a_1 z^{n-1} + a_2 z^{n-2} + \cdots + (-1)^n a_n = 0,$$

with real coefficients $a_1, a_2, \ldots, a_n$, and $\Delta(\bar{a}), \bar{a} = (a_1, a_2, , ..., a_n)$ be its discriminant. In every one-connected domain of a view $D = \{\bar{a} || \Delta(\bar{a}) \neq 0\}$, in which the discriminant is distinct from zero and there exists a point $\bar{a}_0 \in D$ such that the algebraic equation with the such coefficients has exactly s real roots, given equation defines exactly s real smooth implicit functions of coefficients defined everywhere in D, serving as roots of the given equation.

Befor moving forward, we must make some important remarks. The coefficients of characteristic equation, mentioned above, are continuous functions of variables $\bar{u}$. We can implicitly calculate these coefficients using following lemma (see [10]).

Lemma 2. *Let $A \in M_n(\mathbf{R})$ be some matrix of order n. Then singular numbers of this matrix are solutions (which are real) of the algebraic equation*

$$\lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} - \cdots + (-1)^n s_n = 0, \tag{2}$$

where s_m *denotes the sum of all possible Gram determinants of lines of the matrix with all tuples* $(i_1, ..., i_m)$ *such that* $1 \leq i_1 < \cdots < i_m \leq n$.

This lemma is an easy consequence of the theorem 2.1.2 [10] applied to the matrix $A \cdot {}^tA$. Really, the last matrix is a symmetric matrix the entries of which are the scalar product of lines of the matrix A. Then the sum of principle minors of the matrix $A \cdot {}^tA$ with fixed order placed at taken lines is a determinant (which is a Gram determinant) of symmetric matrix containing scalar product of these lines. Combining all lines with one and the same number, we get the demanded result.

One can easily observe that the coefficient s_m is equal to the sum of squares of all minors of order m contained by the matrix A. The result of Lemma 2 unchanged for all matrices of the size $n \times m$. If the rank of the matrix is equal to r, $r \leq n$, then $s_m = 0$ for every $m > r$ and $s_m > 0$ for every $m \leq r$.

Lemma 3. *Singular numbers of the matrix* $G(\bar{u})$ *as the roots of the characteristic equation are differentiable functions of* $\bar{u} \in U$.

Proof. If the discriminant of the characteristic equation is distinct from zero, then singular numbers are pair wisely different. We suppose that the discriminant of the equation is distinct from zero in $U' \subset U$. In consent with Lemma 1, singular numbers are smooth functions in U'. Jacobian matrix of the system of singular numbers is distinct from zero in U'. Moreover, as it is best known (see [3-4]), in imposed conditions, the map $\Lambda : \bar{u} \mapsto \bar{\lambda}(\bar{u})$ defining singular numbers is one-to one, in some neighborhood of given point.

Consider the closed subset of points in which discriminant of the characteristic equation (2) vanishes denoting this set by W. It means that at every point of the set W there are equal singular numbers. We can dissect this set into several number of subsets of a view

$$W_{i,j} = \{\lambda_1, \lambda_2, ..., \lambda_n\},$$

in which $\lambda_i = \lambda_j$ and all other singular numbers are pair wisely distinct. Let Z_i denote the open set of points $(\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1} ..., \lambda_n) \in \mathbf{R}^{n-1}$ (the number λ_i is absent). These numbers are the solution some equations

$$\lambda^{n-1} - b_1\lambda^{n-2} + \cdots + (-1)^{n-1}b_{n-1} = 0,$$

with non-zero discriminant, when $(b_1, ..., b_{i-1}, b_{i+1} ..., b_n)$ varies in an open set U_i. Lemma 1 shows that every point of the set Z_i is a smooth function of coefficients $(b_1, ..., b_{i-1}, b_{i+1} ..., b_n)$. Adding the number $\lambda_i = \lambda_j$, we obtain the set

$W_i = \{(\lambda_1, ..., \lambda_{i-1}, \lambda_i, \lambda_{i+1} ..., \lambda_n) \,|\, (\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1} ..., \lambda_n) \in Z_i\}$. Changing the index i, we find:

$$W_i = \bigcup_{j \neq i} W_{ij}.$$

This union consists of boundary points of the set U'. But projection of every set W_i into $\mathbf{R}^{n-1}$ is an open set.

By an analogy we can consider the sets W_{ijk} in which $\lambda_i = \lambda_j = \lambda_k$, and get a representation

$$W = \bigcup_i W_i = \bigcup_{j \neq i} W_{ij} = \bigcup_{j \neq i \neq k} W_{ijk};$$

every set of a view $W_{i_1,...,i_s}$ has an open projection into the space $\mathbf{R}^{n-s}$. Every subset of this kind consists of points for which some $n - s$ components are pair wisely distinct and othes coinsides with some of them. When s=n, the set $W_{i_1,...,i_s}$ reduces to the fixed points. Prove now that singular numbers are differentiable functions. We have proved this statemnat at such $\bar{u}$ for which the discriminant of the equation (2) is distinct from zero.

Consider the case when the discriminant of the mantioned equation is equal to zero. It is enough to consider subsets of a view W_i, because for other subsets this proof can be easily modified. Let the discriminant of the characteristic equation (2) be equal to zero at the point $\bar{u} \in W$. Vanishing of discriminant means that some singular numbers are equal. For example, suppose that the singular numbers λ_1 and λ_2 are equal: $\lambda_1 = \lambda_2$. Performing Euclid's algorithm, we can find g.c.d of characteristic polynomial $C(\lambda) = C(\lambda; \bar{u})$ in the left hand side of the equation and its derivative, that is, g.c.d($C(\lambda)$,$C'(\lambda)$). Taking ratio $C(\lambda)$/g.c.d($C(\lambda)$,$C(\lambda)$), we find a polynomial without repeated roots. So, the discriminant of got polynomial is distinct from zero.

The procedure of Euclid's algorithm is a procedure of division with a reminder which is possible to perform, if the heading coefficient of the divisor is non-zero. This procedure is finite and reduces to the solution of simlest linear equations over the coefficients, then this procedure can be performed in some neighborhood of a given point $\bar{u}$, due to contiuity of the coefficienrs of the characteristic equation. Since $\bar{u}$ is an arbitrary point of the set W, which is compact, then there are only finite number of neighborhoods in which Euclid's algorithm is possible to perform. Supposing that the singular numbers λ_1 and λ_2 are equal, the polynomial $C_1(\lambda) = C(\lambda)$/g.c.d($C(\lambda)$,$C(\lambda)$) has a degree *n-1*. Then by the theorem ([4]), in this neighborhood the roots of the equation $C_1(\lambda) = 0$, that is, singular numbers $\lambda_2, ..., \lambda_n$ are implicit functions of coefficients. Since $\lambda_1 = \lambda_2$, the correspondence $\bar{u} \to (\lambda_1, \lambda_2, ..., \lambda_n)$ defines a manifold of dimension *n-1* in n-dimensional space.

By this reason all of singular numbers are continuously differentable functions. Lemma 3 is proved.

Let us call the map $\Lambda : \bar{u} \mapsto \bar{\lambda}(\bar{u})$ as spectral, and use non-bolded letters for notating. Denote the Jacobian matrix of this map using the sign of a prime as follows: $\Lambda'(\bar{u}) = (\partial\lambda_i/\partial u_j)$ (this matrix is existing due to Lemma 3). Denote by Φ' the Jacobian matrix of the map $\varphi : U \to \mathbf{R}^n$. Suppose that this map maps an orthonormal basis in U to an orthonormal one in tangential subspace $\mathbf{T}_\varphi \subset \mathbf{R}^n$. Then the Jacobian matrix corresponding to the image N of this map will be of form $F' = \Lambda'\Phi'$. Let us replace the elements of all columns of the matrix F consequently in a line and take the Jacobian matrix of got system of functions, denoting it as F_1.

Theorem 1. *Let in the space $\mathbf{R}^n$ a smooth manifold M be given by the equalities* (1) *and the matrix G_1 is non-singular in U. The the spectral map Λ is one-to one on some neighborhood of the point $\bar{u}$; moreover, following metric relations are true:*

$$\det(F_1 \cdot {}^tF_1) = \det(G_1 \cdot {}^tG_1) = \det\left(\Lambda' \cdot {}^t\left(\Lambda'\right)\right).$$

As it is seen from these equalities the coeffisients of connection does not affect these equalities, independent of the points at which they are taken.

Proof. Suppose first that the spectral map is one-to one. Then in some neighborhood of the point $\bar{u}$ the Jacobian matrix Λ' is non-singular. Consider singular value decomposition of the matrix $G(\bar{u})$ by using of singular bases:

$$G(\bar{u}) = {}^tQ\Lambda T,$$

where Q and T are orthogonal functional matrices of order m, and Λ is a diagonal matrix containing on the diagonal singular number of the matrix $G(\bar{u})$ (see [3]). The columns of the matrices Q and T set up, so called, singular bases of the matrix $G(\bar{u})$. Following relations are best known (see [2]):

$$\lambda_j(\bar{u}) = (G\bar{t}_j, \bar{q}_j),\ G\bar{t}_j = \lambda_j\bar{q}_j, G^T\bar{q}_j = \lambda_j\bar{t}_j. \tag{3}$$

Consider the map $\Lambda : U \to \mathbf{R}^m$. Singular numbers of the matrix $G(\bar{u})$ are the roots of the characteristic equation, which are differentiable. Explicit expessions for the coefficients of the characteristic equation defined by Lemma 2. The set of points in which this equation is fulfilled is closed. Consider the tuples $(s_1, s_2, ..., s_n)$ which set up by images of coefficients of characteristic equation (which are functions of $\bar{u}$). We can distribute the set of all such tuples into the subsets defined in the proof of Lemma 3.

Let us differentiate the relation (2) with respect to any differentiable vector field $\bar{\xi} = \bar{\xi}(\bar{u})$. For this, we take in the relation (1) $\bar{h} = h\bar{\xi}$, and pass to the limit as $h \to 0$. From (2) we can write:

$$D_{\bar{\xi}}\lambda_j(\bar{x}) = (D_{\bar{\xi}}G\bar{t}_j, \bar{q}_j) + (GD_{\bar{\xi}}\bar{t}_j, \bar{q}_j) + (G\bar{t}_j, D_{\bar{\xi}}\bar{q}_j) =$$

$$= (D_{\bar{\xi}}G\bar{t}_j, \bar{q}_j) + \lambda_j(D_{\bar{\xi}}\bar{t}_j, \bar{t}_j) + \lambda_j(\bar{q}_j, D_{\bar{\xi}}\bar{q}_j).$$

Since columns of the matrices Q and T are the unite vectors, then $(\bar{t}_j, \bar{t}_j) = 1$. So, for any vector of
the field $\bar{\xi}$

$$D_{\bar{\xi}}(\bar{t}_j, \bar{t}_j) = 2(D_{\bar{\xi}}\bar{t}_j, \bar{t}_j) = 0 \Rightarrow (D_{\bar{\xi}}\bar{t}_j, \bar{t}_j) = 0.$$

Therefore,

$$D_{\bar{\xi}}\lambda_j(\bar{x}) = D_{\bar{\xi}}(G\bar{t}_j, \bar{q}_j) = ((D_{\bar{\xi}}G)\bar{t}_j, \bar{q}_j), \tag{4}$$

where the symbol $D_{\bar{\xi}}G$ denotes the matrix (derivative in the direction $\bar{\xi}$):

$$D_{\bar{\xi}}G(\bar{u}) = \begin{pmatrix} \frac{\partial^2 g_1'}{\partial u_1 \partial \bar{\xi}} & \frac{\partial^2 g_2'}{\partial u_1 \partial \bar{\xi}} & \cdots & \frac{\partial^2 g_m'}{\partial u_1 \partial \bar{\xi}} \\ \frac{\partial^2 g_1'}{\partial u_2 \partial \bar{\xi}} & \frac{\partial^2 g_2'}{\partial u_2 \partial \bar{\xi}} & \cdots & \frac{\partial^2 g_m'}{\partial u_2 \partial \bar{\xi}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1'}{\partial u_m \partial \bar{\xi}} & \frac{\partial^2 g_2'}{\partial u_m \partial \bar{\xi}} & \cdots & \frac{\partial^2 g_m'}{\partial u_m \partial \bar{\xi}} \end{pmatrix}.$$

From these equalities we note that the vectors of singular bases are remaining constant when we differentiate the singular numbers of the matrix, that is, the coefficients of Cristoffel not taking part in these relations. The vectors of singular basis behave themselves as constant vectors, independent of the field, along which differentiation is taken. We now deduce from (4), for the transposed Jacobian matrix of the map $\Lambda : \bar{u} \mapsto \bar{\lambda}(\bar{u})$, in which ${}^t\bar{\lambda}(\bar{x}) = (\lambda_1(\bar{x}), ..., \lambda_m(\bar{x}))$, a following representation:

$$\frac{\partial(\lambda_1, ..., \lambda_m)}{\partial(u_1, ..., u_m)} = (((D_{u_i}G)\bar{t}_j, \bar{q}_j)) =$$

$$= \begin{pmatrix} ((D_{u_1}G)\bar{t}_1, \bar{q}_1) & ((D_{u_1}G)\bar{t}_2, \bar{q}_2) & \cdots & ((D_{u_1}G)\bar{t}_m, \bar{q}_m) \\ ((D_{u_2}G)\bar{t}_1, \bar{q}_1) & ((D_{u_2}G)\bar{t}_2, \bar{q}_2) & \cdots & ((D_{u_2}G)\bar{t}_m, \bar{q}_m) \\ \vdots & \vdots & \ddots & \vdots \\ ((D_{u_m}G)\bar{t}_1, \bar{q}_1) & ((D_{u_m}G)\bar{t}_2, \bar{q}_2) & \cdots & ((D_{u_m}G)\bar{t}_m, \bar{q}_m) \end{pmatrix}.$$

This matrix has a complicated view. To simplify it, we transform this matrix using special tensor field connected with the matrix $G(\bar{u})$. Consider for this purpose the mapping $\mathbf{\Lambda}_1 : \mathbf{R}^m \to \mathbf{R}^{m^2}$ as a vector function with components being elements of the matrix $G(\bar{u})$: arranging elements of each column of the matrix $G(\bar{u})$ consequently in a line, take the transposed Jacobian matrix of the following system:

$$\mathbf{\Lambda}_1(\bar{u}) = \left(\frac{\partial^2 g_1'}{\partial u_1 \partial u_1}, ..., \frac{\partial^2 g_1'}{\partial u_1 \partial u_m}, \frac{\partial^2 g_2}{\partial u_1 \partial u_1}, ..., \frac{\partial^2 g_2'}{\partial u_1 \partial u_m}, ..., \frac{\partial^2 g_m'}{\partial u_1 \partial u_1}, ..., \frac{\partial^2 g_m'}{\partial u_1 \partial u_m} \right).$$

We have defined a field which can be considered as a tensor field given in $\mathbf{R}^m \otimes \mathbf{R}^m$. Taking Jacobian matrix, we get the matrix with entries

$$\frac{\partial^3 g_i}{\partial u_k \partial u_r \partial u_s}.$$

Conditionally, we denote this matrix as $\mathbf{\Lambda}_1' : \mathbf{R}^m \to \mathbf{R}^{m^3}$. Then we can rewrite the elements of the matrix $\mathbf{\Lambda}_1'$ as below:

$$((D_{u_j} G)\bar{t}_i, \bar{q}_i) = \left((D_{u_j} G)(\bar{t}_i \otimes \bar{q}_i)\right).$$

Moreover, the symbol ${}^t(\bar{t}_i \otimes \bar{q}_i)$ denotes the tensor (direct) product $(t_{i1}q_{i1}, ..., t_{i1}q_{im}, ..., t_{im}q_{i1}, ..., t_{im}q_{im})$. Therefore, the transposed Jacobian matrix$\Lambda'(\bar{u})$ of the mapping $\Lambda : \bar{u} \mapsto \bar{\lambda}(\bar{u})$ is possible represent as follows:

$$\Lambda'(\bar{u}) = \begin{pmatrix} D_{u_1} G' \\ \vdots \\ D_{u_m} G' \end{pmatrix} (\bar{t}_1 \otimes \bar{q}_1 \cdots \bar{t}_m \otimes \bar{q}_m) = G''(T * Q); \; G'' = \begin{pmatrix} D_{u_1} G' \\ \vdots \\ D_{u_m} G' \end{pmatrix}. \quad (5)$$

where the symbol $T * Q$ we use for the matrix containing columns $\bar{t}_1 \otimes \bar{q}_1, ..., \bar{t}_m \otimes \bar{q}_m$. As in the work [1], we call the linear span of these vectors as the principle subspace of the tensor product $Sp(T) \otimes Sp(Q)$ of subspaces $Sp(T)$ and $Sp(Q)$ generated with singular bases. Note that the principle subspace depends on taken singular bases (below we shall describe it implicitly).

The representation (5), got for the Jacobian matrix

$$\frac{\partial(\lambda_1, ..., \lambda_m)}{\partial(u_1, ..., u_m)}$$

considered above, is more appropriate for our carrying out the proof of the theorem. At first, for this purpose, we shall express the quantity $\det \Lambda'(\bar{u})$ using best known integral representation (see [5, p. 131]):

$$(\det \Lambda'(\bar{u}))^{-1} = \pi^{n/2} \left(\Gamma(n/2)\right)^{-1} \int_{\|\Lambda'(\bar{u})\bar{v}\| \le 1} dv_1 \cdots dv_m, \quad (6)$$

where the symbol $\|\cdot\|$ denotes Euclidean norm of the vector. Using representation (5) we can observe that

$$\Lambda'(\bar{u})\bar{v} = \begin{pmatrix} D_{u_1}G' \\ \vdots \\ D_{u_m}G' \end{pmatrix} (\bar{t}_1 \otimes \bar{q}_1 \cdots \bar{t}_m \otimes \bar{q}_m)\bar{v} = G''\left((T * Q)\bar{v}\right).$$

Consider the vector

$$\bar{w} = \bar{V}(\bar{v}) = (T * Q)\bar{v} = v_1\bar{t}_1 \otimes \bar{q}_1 + \cdots + v_m\bar{t}_m \otimes \bar{q}_m. \tag{7}$$

It is not difficult to note that the matrix $T * Q$ has orthogonal normal columns. The relation (7) defines the linear surface given by parametric representation:

$$w_{ij} = v_1 t_{i1} q_{1j} + \cdots + v_m t_{im} q_{mj} i = 1, ..., m;\ j = 1, ..., m. \tag{8}$$

Take m elements of this system of functions:$w_{i_1 j_1}, \ldots, w_{i_m j_m}$. It is easy to observe that the Jacobian matrix of this system seemed as follows:

$$\begin{pmatrix} t_{1i_1}q_{1j_1} & \cdots & t_{mi_1}q_{mj_1} \\ \vdots & \ddots & \vdots \\ t_{1i_m}q_{1j_m} & \cdots & t_{mi_m}q_{mj_m} \end{pmatrix}.$$

The matrix $T*Q$ has $C^m_{m^2}$ number of sub matrices of order m. The sum of squares of such matrices is equal to $\det\left((T * Q) \cdot {}^t(T * Q)\right)$, which equals to 1, because columns of the matrix $T * Q$ are orthonormal. So, the surface element of the linear manifold given by formulae (8) is as follows:

$$ds = \sqrt{\sum \overset{2}{\det} \begin{pmatrix} t_{1r_i}q_{1s_1} & \cdots & t_{mr_1}q_{ms_1} \\ \vdots & \ddots & \vdots \\ t_{1r_m}q_{1s_m} & \cdots & t_{mr_m}q_{ms_m} \end{pmatrix}} dv_1 \cdots dv_m,$$

where the summation is taken over all possible systems of tuples $(r_1, ..., r_m; s_1, ..., s_m) 1 \le r_i \le m, 1 \le s_j \le m$. Sum under the sign of square root is a sum taken over all minors of matrix of order m. Then the surface element is equal to

$$\sqrt{\det\left({}^t\,(T * Q) \cdot (T * Q)\right)}d\bar{v}.$$

So, the equality (6) is possible represent as a surface integral taken over the linear surface introduced above. Therefore,

$$(\det \Lambda'(\bar{u}))^{-1} = \pi^{n/2}\left(\Gamma(n/2)\right)^{-1} \int_{\|G''\bar{V}\| \le 1} ds, \tag{9}$$

where the surface integral taken over the surface (7) and the vector $\bar{V}$ denotes the vector with components w_{ij}. The matrix G'' has the size $n \times n^2$ and has a singular value decomposition of the form:

$$G'' = P\Sigma R$$

in which the matrices P and R are orthogonal matrices of the size n and n^2, correspondingly; the matrix Σ, of the size $n \times n^2$, has a diagonal view, and on the diagonal stay singular numbers of the matrix G''. Do not destroying the generality, we can suppose that all singular numbers are placed on first m columns of the matrix Σ.

Let us make exchange of variables of a view

$$\bar{W} = R\bar{V}$$

under the integral. We have

$$\left\|G''\bar{V}\right\|^2 = (P\Sigma R\bar{V}, P\Sigma R\bar{V}) = (\Sigma^2 R\bar{V}, R\bar{V}) = (\Sigma^2\bar{W}, \bar{W}).$$

Since transformation R is non-singular, then the subspace generated by images of the column-vec-tors of the matrix $T * Q$ is also m-dimensional. After changing of variables, in the right hand side of (9), these variables stand variables of integration, m of which are independent, because they define the subspace of m dimension. Let the matrix $R(T * Q)$, containing images of the columns of the matrix $T * Q$, to have a basic minor containing rows of the matrix $R(T * Q)$ with numbers$1 \leq j_1, ..., j_m \leq m^2$. From our assumption above, we deduce:

$$(\det \Lambda'(\bar{u}))^{-1} = \pi^{n/2} \left(\Gamma(n/2)\right)^{-1} \int_{\left\|\sigma_1^2 w_1 + \cdots + \sigma_m^2 w_m\right\| \leq 1} d\bar{w},$$

and the integral is convergent when and only when the first m rows of the matrix $R(T * Q)$ are linearly independent. In this case variables $w_1, ..., w_m$ are independent. Otherwise, (that is, there is not existing such a basic minor) they are not independent and at least one of them, say the last addend in the conditions under the sign of surface integral, is of form

$$\sigma_m^2(c_1 w_1 + \cdots + c_{m-1} w_{m-1}).$$

Let us clarify this statement. From (8) it follows that the components of the vector ${}^t\bar{w} = (w_1, ..., w_m)$ are the first m components of linear combination of vector columns of the matrix $T * Q$. Since first m rows are linearly dependent, then corresponding sums $w_{ij} = v_1 t_{i1} q_{1j} + \cdots + v_m t_{im} q_{mj}$ in (8) is not univaludely defining variables $w_1, ..., w_m$. So, these variables are dependent. By this reason, at least one of integrating variables, say the last variable w_m defining the linear span of columns of the matrix $R(T * Q)$, does not take part in the condition under the sign of surface integral, defining the domain of integration. Therefore, the surface integral over m dimensional subspace is divergent, due to existence of additional mute independent variable of integration. This means that the determinant of the matrix $\Lambda'(\bar{u})$ is equal to zero which contradicts our assumption. So, we have:

$$(\det \Lambda'(\bar{u}))^{-1} = \pi^{n/2} \left(\Gamma(n/2)\right)^{-1} \int_{\|\Sigma \bar{W}\| \leq 1} dw = (\sigma_1 \cdots \sigma_m)^{-1} .$$

Since the numbers $\sigma_1, \ldots, \sigma_m$ are singular numbers of the matrix G'', then the integral above equals to

$$\left(\det G'' \cdot {}^t G''\right)^{-1/2} .$$

Consider now the case when spectral map is singular. Let ε be suffisienly small positive number. Take instead of the map Λ the following map $\Lambda_\varepsilon : \bar{u} \mapsto \lambda_\varepsilon(\bar{u})$ and:

$$\lambda_\varepsilon(\bar{u}) = (\lambda_1(\varepsilon), ..., \lambda_n(\varepsilon)); \lambda_i(\varepsilon) = \lambda_i(\bar{u}) + \varepsilon u_i .$$

Then $\Lambda'_\varepsilon = \Lambda' + \varepsilon E$, where E is a unite matrix. Then we arrive at initial case. Take in singular decomposition of the matrix $G(\bar{u})\ \Lambda_\varepsilon$ instead of $\Lambda(\bar{u})$. When ε is sufficiently small, derivative of the got matrix will be non-singular also. Now we have:

$$\det(\Lambda' \cdot {}^t\Lambda') = \det(G'_\varepsilon \cdot {}^t G'_\varepsilon); G_\varepsilon = {}^t Q \Lambda_\varepsilon T.$$

Tending ε to 0, we get required result. Particularly, the sace singularity of spectral matrix impossible in conditions of the theorem. Theorem 1 is proved.

Corollary 1. *Let the manifold defined by the system* (1) *be non-singular in the domain U. Then the map f is non-singular, that is the matrix F_1 has maximal rank, at everi inner point of given manifold, at which Jacobian matrix $\Lambda'(\bar{u}) = (\partial\lambda_i/\partial u_j)$ of the map $\Lambda : \bar{u} \mapsto \bar{\lambda}(\bar{u})$ is non-singular.*

Note that this result does not mean that the singular numbers of the matrices F_1 and $\Lambda'(\bar{u})$ are identical. Singular value decompositions of them is taken in various spaces.

Corollary 2. *The set of singular points of the map* (1) *is a set free from inner points, if the matrix F_1 has maximal rank everywhere in U.*

Really, if the set of singular points has an inner point, then in some neighborhood of this point the equality of Theorem 1 is satisfied. So, we have got a contradiction, that is the Jacobian matrix is non-singular. Therefore the statemnet of the consequaence 2 is true.

Corollary 3. *The set of singular points of the map* (1) *has zero Jordan measure, if the matrix* F_1 *has maximal rank everywhere in* U*.*

Really, the fact that the set of singular points of the map (1) is a Jordan set is shown above in the proof of Theorem 1, using Lemma 3. This Jordan subset, by Consequence 2, consists of boundary points only. Therefore, by the definition of the Jordan set this set has a zero Jordan measure.

4. On some structural questions connected to tensor fields

The prosedure of investigation of singular numbers of smooth maps' Jacobi matrices (briefily, spectral maps' derivatives) given above based on constructing of two tensor fields and studing metric questions connected with them. The prosedure can be well generalized to derivatives of high order. Bypassing difficulties connected with coefficients of connection, we obtain relations between them. We introduce for such fields more appropriate obvious presentation as matrices of several dimensions (cubic, biquadratic, quintic and ets) .

We have seen, that when the matrix F_1 has maximal rank everywhere in U, the linear subspace along which the surface integral is taken, is an isomorphic image of some subspace of the tensor product $\mathbf{R}^m \otimes \mathbf{R}^{m^2}$ generated by some vectors of the form $x \otimes y$, $x \in \mathbf{R}^m$, $y \in \mathbf{R}^{m^2}$. Such a subspace we shall call as principle subspace. We see that at vectors of principle subspace, the matrix F_1 (strictly saying the matrix ${F_1}^t(F_1)$) reduces to the diagonal form. If we reperesent every column of the matrix F_1 as a matrix of order m, we can fancy it as a cubic matrix consisting of m number of quadratic layers. Now we can describe the way for getting of this form as follows. Taking of Jacobian matrix of a row of functions is a procedure substituting of every function by its gradient vector (in perpendicular direction on the plane). We get Jacobian matrix. Now we substitute every function of this matrix by its gradient vector in the direction perpendicular to the plane at which the matrix is placed. We get a cubic "matrix" which is appropriate to call as a "Jacobian matrix of high order" induced by, so called, the gradient of high order. We can visualize the reducing of the matrix F_1 to the diagonal form described above as a procedure reducing every layer to diagonal form to get a matrix placed at diagonal section of the cube, and then reducing this matrix to the diagonal form, to get singular numbers on the diagonal of the cube. Note that the possibility of such reducing is hinted out by the note above about the fact that aroused matrices actually depend on m number of parameters, despite huge dimensions of spaces.

Consider now two sequences of matrices: 1) $\Lambda(\bar{u}), \Lambda_1(\bar{u}), ..., \Lambda_{k-2}(\bar{u})$; 2) $\mathrm{M}_1(\bar{u})$, $\mathrm{M}_2(\bar{u})$,…,$\mathrm{M}_{k-2}(\bar{u})$ constructions of which described below.

1)The matrix $\Lambda(\bar{u}) = G(\bar{u}) = F\Phi$ is a Jacobian matrix constructed above, that is the Jacobian matrix of the system of coordinate functions for the gradient ∇g. $\Lambda_1(\bar{u})$ is a transposed Jacobian matrix of the system of functions being a singular numbers of the matrix $\Lambda'(\bar{u})$, that is the Jacobian matrix of the map. Further, by induction, if the matrix $\Lambda_j(\bar{u})$ is defined then by $\Lambda_{j+1}(\bar{u})$ we denote the transposed Jacobian matrix for the system of singular number of the matrix $\Lambda_j(\bar{u})$.

2) Construction of the second sequence we begin from the matrix $M_0\bar{u}) = \Lambda(\bar{u})$. We take as a matrix $B_1(\bar{u}) = \Lambda'(\bar{u})$ the transposed Jacobian matrix of the system of n^2 functions got arranging the elements of columns, consequently, in a line. So, this a matrix is of the size $n \times n^2$. Further, we arrange the elements of columns of the matrix $M_1(\bar{u})$, consequently, in a row, and take as a matrix $M_2(\bar{u})$ the transposed Jacobian matrix of the got system of functions. Continuing by such way, we get the sequence of matrices $B_1(\bar{u})$,…,$B_{m-2}(\bar{u})$. The matrix $B_{m-2}(\bar{u})$ consists of all partial derivatives of the function $g(\bar{u})$.

Theorem 2. *Let $m \geq 3$, and $0 \leq k \leq m-3$, If $\det(M_{k+1} \cdot {}^tM_{k+1}) \neq 0$ for all $\bar{x} \in U$ then the relation below holds: $(\det \Lambda_k(\bar{u}))^2 = \det(M_k \cdot {}^tB_k)$.*

Proof of this theorem is an easy consequence of Theorem 1, applied to the Jacobian matrices of spectral maps.

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