RAIRO-Oper. Res. 59 (2025) 1443–1473
https://doi.org/10.1051/ro/2025042

RAIRO Operations Research
www.rairo-ro.org

QUEUEING-INVENTORY SYSTEM WITH RETURN OF PURCHASED ITEMS AND CUSTOMER FEEDBACK

Dhanya Shajin[1,2,*] and Agassi Melikov[3]

Abstract. In this paper, a model of single server queueing-inventory system (QIS) with Markovian Arrival Process (MAP) and phase-type distribution (PH-distribution) of the service time of consumer customers (c-customers) is considered. After completing the service of c-customer, he (she) can make one of the following decisions: (1) eventually leave the system with probability (w.p.) σ_ℓ; (2) after a random "thinking" time returns the purchased item w.p. σ_r; (3) after a random "thinking" he (she) feedback to buy a new item w.p. σ_f. It is assumed that $\sigma_\ell+\sigma_r+\sigma_f=1$. If upon arrival of the c-customer the system main warehouse (SMW) is empty, then the incoming customer, according to the Bernoulli scheme, is either joins the infinite queue or leaves the system. A virtual finite orbit can be considered as a waiting room for feedback customers (f-customers). Returned items are considered new and are sent directly to SMW if there is at least one free space; otherwise, this item is sent to a special warehouse for returned items (WRI). After completing the service of each customer, one item is instantly sent from the WRI (if any) to the SMW. In SMW, the (s,S) replenishment policy is used and it is assumed that the lead time follows exponential distribution with finite parameter. When the stock level reaches its maximum value due to items returns, the system immediately cancels the regular order. Along with classical performance measures of QIS new specific measures are defined and numerical method for their calculation as well as maximization of the revenue function are developed. Results of numerical examples to illustrate the effect of different parameters on the system's performance measures are provided and analyzed. We also provide a detailed analysis of an important special case of the Poisson process/exponential service time model.

Mathematics Subject Classification. 60J28, 60K25, 90B05, 90B22.

Received June 29, 2024. Accepted April 7, 2025.

1. Introduction

The pioneering works in the theory of queuing-inventory systems (QISs) are those of Sigman and Simchi-Levi [43] and Melikov and Molchanov [30] which were published independently each other. After these works, in last three decades, this theory has become an independent and rapidly developing part of operations research. Numerous papers published over the years have taken into account various specific features of QIS models. So,

Keywords. Queueing-inventory, returning items, feedback, Markovian arrival process, phase type distribution.

[1] Department of Mathematics, T K Madhava Memorial College, Nangiarkulangara, Alappuzha 690513, Kerala, India.

[2] Department of Mathematics, Sanatana Dharma College, Alappuzha 688003, Kerala, India.

[3] Baku Engineering University, Khirdalan, Hasan Aliyev Str., 120 AZ 0101, Azerbaijan.

*Corresponding author: dhanya.shajin@gmail.com

the models of perishable and non-perishable QISs under different replenishment policies have been studied by various authors, see fundamental studies by Daduna [14], Otten and Daduna [33], Schwarz and Daduna [36], Schwarz *et al.* [37], Krishnamoorthy and Jose [22], Chakravarthy and Rao [11], Chakravarthy and Rumyantsev [12], Jose and Nair [20], Baek [8], Rasmi *et al.* [34] etc.

It is obvious that within the framework of one model it is not possible to take into account all kinds of specific features of QISs. One of the important features of the systems is the return of purchased items. Indeed, returning of a purchased inventory due to various reasons (*e.g.* turns out to be faulty or low-quality items, doesn't look or work as advertised) before its expiry is a common phenomenon in the real-life queuing-inventory systems. For example, under EU rules, a seller has the right to a minimum 2-year guarantee, at no cost, and he (she) can return any purchase within 14 days without justification (well-known 14-day cooling-off period, that is, a period of time following a purchase when the customer may choose to cancel a purchase, and return items which have been supplied, for any reason, and obtain a full refund). This phenomenon has been studied in detail for QIS with common life time (CLT) inventory by Shajin *et al.* [40], Krishnamoorthy *et al.* [24, 25], Krishnamoorthy *et al.* [26], Shajin *et al.* [41], Shajin and Krishnamoorthy [39], Shajin *et al.* [42]. In these papers, seats on scheduled transport (*e.g.* flight/train/bus) are considered as CLT inventory. Indeed, once the scheduled transport departs, all inventory (sold and unsold), is lost because the life time of the items in the inventory has expired.

A review of works devoted to the study of classical inventory management systems (*i.e.* in systems without a service station) taking into account the effect of the return of purchased items can be found in Marisa *et al.* [15]. But the influence of this effect was not taken into account in the models of queuing-inventory systems.

Let's consider a brief overview of work on QIS models with CLT, which take into account the return of purchased inventory. The pioneer papers in this topic was done by Krishnamoorthy *et al.* [24, 25]. Authors considered single-server QIS with exponentially distributed CLT in which arrived customer immediately taken for service if on arrival the server is idle and the inventory level is positive; otherwise, he (she) is added to a limited buffer, the size of which depends on inventory level. Consumers form a homogeneous Poisson process, and their service time in the server has an exponential distribution. If the waiting room is completely full when a customer arrives, then the arriving customer goes into an orbit of infinite capacity or is lost forever according to Bernoulli's trials. The intensity of retrial customers from the orbit is a constant value. The return of sold goods occurs according to an exponentially distributed time, where its parameter depends on the level of inventory. It is assumed that lead time is zero, *i.e.* on expiry of CLT, the inventory reaches its maximum level instantaneously for the next cycle. Customers being pushed out from the finite waiting room, but not from the orbit, when the CLT is implemented. Main performance measures are calculated and numerical examples are demonstrated, including the results of some optimization problems. Similar model was analyzed in Krishnamoorthy *et al.* [26]. In this paper, the authors show that if no customers join the system when the inventory level is zero, the product form solution for the distribution of system states is guaranteed. Model of QIS in which the reservation has to be done in one of the multiple time frames (slots) was considered by Shajin *et al.* [41]. At the beginning of each time frame, the inventoried items have CLT with phase-type distribution and the customer arrivals forms a Markovian arrival process (MAP). Unlike previous papers, reservations through overbooking are allowed here (for each time frame), if at the time of the start of customer service the inventory is not available in the required period of time. All overbooked customers present in the recently expired time slot will be granted a reservation in the newly added time slot. Similar model was considered in Shajin and Krishnamoorthy [39]. Here the authors show that in the special case of Poisson process/exponentially distributed service time there is an asymptotic solution in product form. In Shajin *et al.* [42], a correlated QIS with (s, S) replenishment policy, MAP flow, PH-distributed service time and exponential replenishment time was investigated. Items in each cycle have a CLT with an exponential distribution, and before this is realized, a purchased item in a cycle can be canceled in that cycle itself, provided that inventory levels have not dropped to zero. The time between cancellations follows independent exponential distributions. For the special case of Poisson process/exponential service time, authors show that a product form solution is existing, *i.e.* extending previous work to the case of correlated lead time is performed. At the end of this paragraph note that models of QISs with CLT without returning of a purchased inventory appeared earlier in the inventory literature in papers by Lian and Neuts [28] and Chakravarthy [10] and continues to be researched to this day, see Dissa and Ushakumari [16,17].

All customers leaving the QIS can be divided into three groups: (1) customers who permanently leave the system and will not return (episodic customers); (2) customers who are thinking about returning purchased inventory (hesitant customers) and (3) customers who will return for a new part of the inventory (regular customers). Here we take into account all three types of customers, *i.e.* we study a real-life case which combines returning of a purchased goods before the expiry of specified time (this time is determined by the laws of each country) and feedback of satisfied customers. Returning purchased inventory in QIS with non-perishable inventory as well as individual life time (ILT) inventory does not appear to have been previously discussed in the available literature. Krishnamoorthy *et al.* [23], Krishnamoorthy *et al.* [27] and Choi *et al.* [13] provides a detailed review on work in models of QIS with positive service time. These reviews do not contain any reference related to returning of a purchased inventory. Since these reviews, several papers have been published that consider the return of purchased inventory, see Vinitha *et al.* [47], Jeganathan *et al.* [19] and Saranya *et al.* [35]. Note that these papers make some unrealistic for QISs assumptions. Thus, Jeganathan *et al.* [19] do not take into account the limited size of the warehouse, that is, the classical inventory system was considered. Vinitha *et al.* [47] consider a system with instant service (*i.e.* service time is zero) and assume that the system allows returns of purchased items only if the warehouse is not full. Saranya *et al.* [35] assumed that returned goods are stored in a warehouse of unlimited capacity. In this paper, we excluded unrealistic assumptions, that is, here we develop a QIS model that is more adequate to the real situation.

In order to reduce the number of returns of sold items due to poor quality, in some cases an effective solution may be to organize a quality check (inspection) of items arriving at the warehouse. In recent years, models of such QISs have been studied in the works of Aghsami *et al.* [1,2].

In QISs, in addition to primary customers, repeat customers are often also serviced. It is necessary to distinguish between two types of repeat customers: (1) customers that require re-service because there was either no space in the system, no inventory, or both when they were received; (2) customers that have already been serviced previously and require re-service due to a number of reasons. The first type of systems was considered by Manikandan and Nair [29], where the authors considered a single sever QIS model $M/M/1/1$ in which failed attempts to access an idle server are joined to an orbit of infinite capacity. Lost sale scheme is used in the system, *i.e.* primary customers, who encounter an idle server without stock at its arrival epoch, leave the system for ever. The condition for stability is obtained and an algorithmic approach for the computation of the system steady-state probabilities and performance measures is used. The interested reader can find further references in this area in Shajin and Krishnamoorthy [38]. Recently, Hanukov [18] proposed a new retrial scheme where the physical presence of the customer in the system is not required during the service period in the server, *i.e.* the customer can go into orbit, and then contact the system after a random time to determine whether the service of its order has been completed or not; if the order has not yet been completed, the customer is sent back into orbit.

Systems of the second type are called feedback systems. "The terminology feedback" first appeared in the queuing literature in the work of Takacs [45,46]. The literature on queuing with feedback is very rich, see, for instance, Koroliuk *et al* [21], Ayyapan and Karpagam [6], Ayyapan and Thilagavathy [7] and Bouchentouf *et al.* [9]. However, it is only recently that this terminology has appeared in the QIS literature in papers by Amirthakodi and Sivakumar [3,4], Amirthakodi *et al.* [5], Suganya *et al.* [44] and Melikov *et al.* [31]. The feedback (instantaneous or delayed) is divided into single feedback (a customer can feedback only once) and multiple feedback (where a customer can feedback several times). In QIS, feedback can occur for two reasons: (1) when the customer's previous service is not satisfactory (for example, in communications networks, erroneously transmitted packets require retransmission) and (2) when the customer is very satisfied with the previous service and therefore returns to the system (in shopping centre, customers who receive pleasant service return to this centre). Here we assume that in both cases the feedback customer will purchase the inventory but will not return the purchased inventory. In other words, in this paper we consider a real-life case which combines items return before the specified time of expiry and feedback satisfied customers. Summarize, the main novelty of the proposed model lies as it simultaneously takes into account two opposite phenomenon in real QISs: return of items sold from unsatisfied customers and purchase of a new batch of items from satisfied customers. Moreover, both events are not instantaneous, that is, the decision to return the sold item and feedback is implemented during some (random) delay after completion of the previous service.

The main contributions of our paper are as follows:

– A new and adequate to the real situation QIS model is proposed, which takes into account the returns of items sold from dissatisfied customers, as well as feedback from already served satisfied customers for the purchase of a new batch of items.
– We study a QIS model with a MAP flow, PH-distributed service time, exponential lead time and two warehouses, one of which is assigned to items from an external source, and the other warehouse is intended for storing returned items.
– The stability condition of the constructed mathematical model of the studied QIS was obtained and algorithms for calculating the steady-state probabilities and the desired performance measures were developed.
– For the Poisson/exponential model, it is shown that in the case of a lost sale scheme, the stability condition depends only on the intensity of primary customers and their service time and does not depend on other parameters of the system.
– Although for the sake of concreteness of the presentation it is assumed that the system uses the (s, S) replenishment policy from an external source, the proposed approach can be used for other policies as well.
– Behavior of the performance measures *versus* the QIS parameters is demonstrated as results of numerical experiments, and the results of the problem of maximization of the revenue are discussed.

Following notations, abbreviations and definitions are needed:

I	: Identity matrix of appropriate order
$\mathbf{e}$	: Column vector of 1's of appropriate order
$\otimes$	: Kronecker product
$\oplus$	: Kronecker sum
QIS	: Queueing inventory system
MAP	: Markovian Arrival Process
CLT	: Common life time
PH	: Phase type distribution
SMW	: System main warehouse
WRI	: Warehouse for returned items
CTMC	: Continuous time Markov Chain
c-customers	: Consumer customers
f-customers	: Feedback customers

S	Main warehouse maximum capacity
ϕ_1	Probability of joins the queue
ϕ_2	Probability of leaves the system
(D_0, D_1)	MAP representation of order a
$(\boldsymbol{\alpha}, T)$	PH representation of order b
σ_r	Probability of returns the purchased item and returns the purchased item after the "thinking" within a random time $\sim \exp(\zeta_r)$
σ_f	Probability of feedback to buy a new item and feedback to buy a new item after the "thinking" within a random time $\sim \exp(\zeta_f)$
σ_ℓ	Probability of eventually leaves the system
s	Reorder point
θ	Positive lead time $\sim \exp(\theta)$
K	Special warehouse for returned items with finite capacity
N	Capacity of finite orbit (waiting space for feedback customers) and inter-occurrence time from the orbit $\sim \exp(\eta)$

The rest of this paper as follows: Section 3 describes the system in general set up and established stability. Steady state analysis is provided in Section 4 and some performance measures investigated. Numerical illustrations are provided in Section 5. The next section analyzes the special case with all underlying distributions exponential. Finally given a few suggestions for future work.

2. Description of the model

Consider a single server queueing-inventory system (QIS) in which main warehouse has maximum capacity S. Input flow of homogeneous consumer customers (c-customers) follows Markovian arrival process with representation (D_0, D_1) of order a. Let $\boldsymbol{\gamma}$ be the steady state probability vector of $D = D_0 + D_1$. Then, $\boldsymbol{\gamma}$ satisfy $\boldsymbol{\gamma}D = 0$ and $\boldsymbol{\gamma}\mathbf{e} = 1$. The fundamental rate is given by $\lambda = \boldsymbol{\gamma}D_1\mathbf{e}$ which gives the expected number of arrivals per unit of time. Customers homogeneity means that each customer requires the same size of inventory. The waiting room for c-customers is infinite size. A hybrid sale scheme is applied, that is, if upon arrival of c-customer the inventory level is zero, then, in accordance to the Bernoulli trials, it either joins the queue with probability ϕ_1 (backorder sale scheme) or leaves the system without inventory with probability $\phi_2 = 1 - \phi_1$ (lost sale scheme). Service time of the c-customer has phase type distribution with representation $(\boldsymbol{\alpha}, T)$ of order b. The mean service of the customer is calculated by $\mu' = -\boldsymbol{\alpha}T^{-1}\mathbf{e}$. At the end of each service the inventory level decreases by one unit.

Once a customer leaving the system after completing his (her) service, he (she) can make one of the following decisions:

- he (she) returns the purchased item with probability σ_r after the "thinking" within a random time. It follows exponential distribution with parameter ζ_r;
- he (she) can feedback to buy a new item with probability σ_f after the "thinking" within a random time which is exponentially distributed with parameter ζ_f;
- he (she) eventually leaves the system with probability σ_ℓ;

where $\sigma_\ell + \sigma_r + \sigma_f = 1$.

The returned item is considered as fresh one and it goes directly to the system main warehouse (SMW) if at least one free space there. If the main warehouse is completely full, then this item is sent to a special warehouse for returned items (WRI) with finite capacity (say) K. Return of item is not allowed when the special warehouse is full, *i.e.* it is assumed that unsuccessful attempts to return an item will be back to its source in case of a full WRI and he/she will try again in the future. After completing service of each customer, one item is instantly sent to the SMW from the WRI (if any). In SMW, the inventory system follows (s, S) policy when the inventory level drops to the reoder point s order placed with positive lead time. It follows exponential distribution with parameter θ. When the inventory level reaches S due to returning of items, the system instantly cancels the regular order.

A virtual finite orbit of size (say) N can be considered as a waiting room for feedback customers (f-customers). Inter-occurrence time for f-customers from the orbit have an exponential distribution with parameter η. Service time distribution of f-customers is same as c-customers. If upon arrival of f-customer server is idle and inventory level is positive, then its service time started, else (that is, at the moment the server is busy or no items in the main warehouse) the f-customer will return to orbit and retry for the service (see graphical description in Fig. 1). It is assumed that the arrival of primary customers, retrial of feedback customers and arrival of customers who return items, service times and replenishment processes are mutually independent.

Define $N_S(t)$ as the number of customers in the system including the one in service, $I_r(t)$, the number of items in the special warehouse (WRI), $N_O(t)$ is the number of feedback customers in the orbit, $I(t)$ is the number of items in the main warehouse, $C(t)$ is the phase of service and $M(t)$ is the phase of arrival process. Then $\Omega = \{(N_S(t), I_r(t), N_O(t), I(t), C(t), M(t)), t \geq 0\}$ is a continuous time Markov chain (CTMC) on the state space

$$\{(0, 0, n_o, i, k), 0 \leq n_o \leq N, 0 \leq i \leq S, 1 \leq k \leq a\}$$

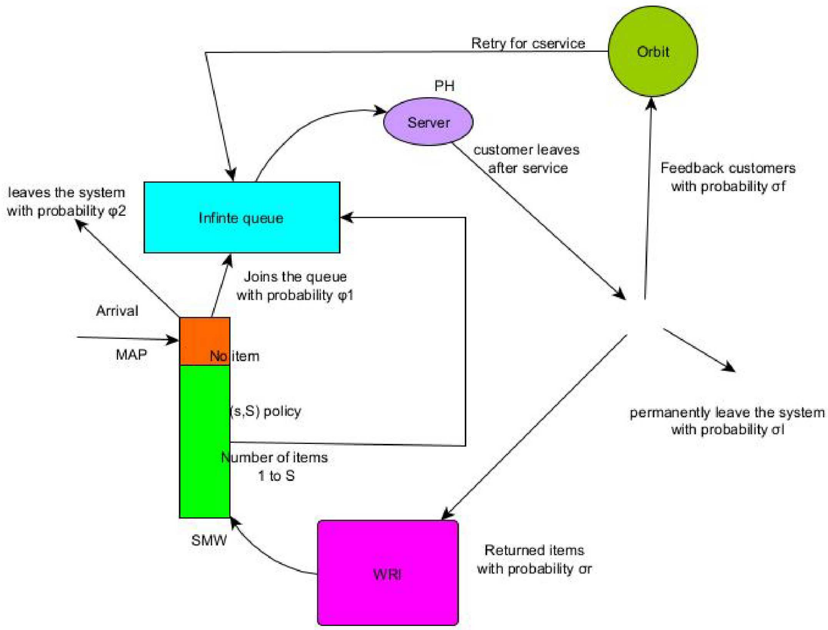

FIGURE 1. Picture representation of the model.

$$\bigcup\{(0,i_r,n_o,S,k), 1\le i_r\le K, 0\le n_o\le N, 1\le k\le a\}$$
$$\bigcup\{(n,0,n_o,0,k), n\ge 1, 0\le n_o\le N, 1\le k\le a\}$$
$$\bigcup\{(n,0,n_o,i,j,k), n\ge 1, 0\le n_o\le N, 1\le i\le S, 1\le j\le b, 1\le k\le a\}$$
$$\bigcup\{(n,i_r,n_o,S,j,k), n\ge 1, 1\le i_r\le K, 0\le n_o\le N, 1\le j\le b, 1\le k\le a\}.$$

The transition rates are:

(a) Transitions rates due to the arrival:

$$\begin{array}{lll}
(n,0,n_o,0)\to(n+1,0,n_o,0): & \text{rate } \ \phi_1 D_1 & \text{for } n\ge 0, 0\le n_o\le N\\
(0,0,n_0,i)\to(1,0,n_0,i): & \text{rate } \ \boldsymbol{\alpha}\otimes D_1 & \text{for } 0\le n_o\le N, 1\le i\le S\\
(0,i_r,n_0,S)\to(1,i_r,n_0,S): & \text{rate } \ \boldsymbol{\alpha}\otimes D_1 & \text{for } 1\le i_r\le K, 0\le n_o\le N\\
(n,0,n_0,i)\to(n+1,0,n_0,i): & \text{rate } \ I\otimes D_1 & \text{for } n\ge 1, 0\le n_o\le N, 1\le i\le S\\
(n,i_r,n_0,S)\to(n+1,i_r,n_0,S): & \text{rate } \ I\otimes D_1 & \text{for } n\ge 1, 1\le i_r\le K, 0\le n_o\le N.
\end{array}$$

(b) Transitions rates due to service completion:

$$\begin{array}{lll}
(1,0,n_o,i)\to(0,0,n_o,i-1): & \text{rate } \ \mathbf{T}^0\otimes I & \text{for } 0\le n_o\le N, 1\le i\le S\\
(1,i_r,n_0,S)\to(0,i_r-1,n_0,S): & \text{rate } \ \mathbf{T}^0\otimes I & \text{for } 1\le i_r\le K, 0\le n_o\le N\\
(n,0,n_0,1)\to(n-1,0,n_0,0): & \text{rate } \ \mathbf{T}^0\otimes I & \text{for } n\ge 2, 0\le n_o\le N\\
(n,0,n_0,i)\to(n-1,0,n_0,i-1): & \text{rate } \ \mathbf{T}^0\boldsymbol{\alpha}\otimes I & \text{for } n\ge 2, 0\le n_o\le N, 2\le i\le S\\
(n,i_r,n_0,S)\to(n-1,i_r-1,n_0,S): & \text{rate } \ \mathbf{T}^0\boldsymbol{\alpha}\otimes I & \text{for } n\ge 2, 1\le i_r\le K, 0\le n_o\le N.
\end{array}$$

(c) Transitions rates due to retrial:

$$\begin{array}{lll}
(0,0,n_o,i)\to(1,0,n_o-1,i): & \text{rate } \ n_o\eta\boldsymbol{\alpha}\otimes I & \text{for } 1\le n_o\le N, 1\le i\le S\\
(0,i_r,n_0,S)\to(1,i_r,n_0-1,S): & \text{rate } \ n_o\eta\boldsymbol{\alpha}\otimes I & \text{for } 1\le i_r\le K, 1\le n_o\le N.
\end{array}$$

(d) Transitions rates due to replenishment:

$$\begin{array}{lll}
(0,0,n_o,i) \to (0,0,n_o,S): & \text{rate } \theta \otimes I & \text{for } 0 \le n_o \le N, 0 \le i \le S-1 \\
(n,0,n_o,0) \to (n,0,n_o,S): & \text{rate } \theta\boldsymbol{\alpha} \otimes I & \text{for } n \ge 1, 0 \le n_o \le N \\
(n,0,n_o,i) \to (n,0,n_o,S): & \text{rate } \theta \otimes I & \text{for } n \ge 1, 0 \le n_o \le N, 1 \le i \le S-1.
\end{array}$$

(e) Transitions rates due to return of items:

$$\begin{array}{lll}
(0,0,n_o,i) \to (0,0,n_o,i+1): & \text{rate } \sigma_r\zeta_r \otimes I & \text{for } 0 \le n_o \le N, 0 \le i \le S-1 \\
(0,i_r,n_o,S) \to (0,i_r+1,n_o,S): & \text{rate } \sigma_r\zeta_r \otimes I & \text{for } 0 \le i_r \le K-1, 0 \le n_o \le N \\
(n,0,n_o,0) \to (n,0,n_o,1): & \text{rate } \sigma_r\zeta_r\boldsymbol{\alpha} \otimes I & \text{for } n \ge 1, 0 \le n_o \le N \\
(n,0,n_o,i) \to (n,0,n_o,i+1): & \text{rate } \sigma_r\zeta_r \otimes I & \text{for } n \ge 1, 0 \le n_o \le N, 1 \le i \le S-1 \\
(n,i_r,n_o,S) \to (n,i_r+1,n_o,S): & \text{rate } \sigma_r\zeta_r \otimes I & \text{for } n \ge 1, 1 \le i_r \le K-1, 0 \le n_o \le N.
\end{array}$$

(f) Transitions rates due to feedback of customers:

$$\begin{array}{lll}
(0,0,n_o,i) \to (0,0,n_o+1,i): & \text{rate } \sigma_f\zeta_f \otimes I & \text{for } 0 \le n_o \le N-1, 0 \le i \le S \\
(n,i_r,n_o,S) \to (n,i_r,n_o+1,S): & \text{rate } \sigma_f\zeta_f \otimes I & \text{for } n \ge 0, 1 \le i_r \le K, 0 \le n_o \le N-1 \\
(n,0,n_o,i) \to (n,0,n_o+1,i): & \text{rate } \sigma_f\zeta_f \otimes I & \text{for } n \ge 1, 0 \le n_o \le N-1, 0 \le i \le S.
\end{array}$$

Thus the infinitesimal generator of Ω is of the form

$$\mathcal{Q} = \begin{pmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1}$$

Each matrix A_0, A_1, A_2 are square matrix of order p and matrices A_{00}, A_{01}, A_{10} are of order $q \times q, q \times p, p \times q$ respectively where $p = (N+1)[(S+K)b+1]a$ and $q = (N+1)(S+K+1)a$ where

$$A_0 = \begin{pmatrix} L_0 & & & \\ & L & & \\ & & \ddots & \\ & & & L \end{pmatrix}, \quad A_2 = \begin{pmatrix} M_0 & & & \\ M_1 & & & \\ & M & & \\ & & \ddots & \\ & & & M \end{pmatrix}, \quad A_1 = \begin{pmatrix} G_0 & G & & & \\ & G_1 & G_2 & & \\ & & \ddots & \ddots & \\ & & & G_1 & G_2 \\ & & & & G_3 \end{pmatrix},$$

$$A_{01} = \begin{pmatrix} \hat{L}_0 & & & \\ & \hat{L} & & \\ & & \ddots & \\ & & & \hat{L} \end{pmatrix}, \quad A_{10} = \begin{pmatrix} \hat{M}_0 & & & \\ \hat{M}_1 & & & \\ & \hat{M} & & \\ & & \ddots & \\ & & & \hat{M} \end{pmatrix}, \quad A_{00} = \begin{pmatrix} \hat{G}_0 & \hat{G} & & & \\ & \hat{G}_1 & \hat{G}_2 & & \\ & & \ddots & \ddots & \\ & & & \hat{G}_1 & \hat{G}_2 \\ & & & & \hat{G}_3 \end{pmatrix}.$$

The sub-matrices are completely defined in Appendix A.

2.1. Stability condition

Let $\boldsymbol{\pi}$ be the steady state probability vector of $A = A_0 + A_1 + A_2$. Then

$$\boldsymbol{\pi} A = \mathbf{0}, \ \boldsymbol{\pi}\mathbf{e} = 1. \tag{2}$$

From (2) we have

$$\boldsymbol{\pi}_0 H_0 + \boldsymbol{\pi}_1 M_1 = 0,$$

$$\begin{gathered}\boldsymbol{\pi}_0 G+\boldsymbol{\pi}_1 H_1+\boldsymbol{\pi}_2 M=0,\\ \boldsymbol{\pi}_{i-1} G_2+\boldsymbol{\pi}_i H_1+\boldsymbol{\pi}_{i+1} M=0,\quad 2\le i\le K-1\\ \boldsymbol{\pi}_{K-1} G_2+\boldsymbol{\pi}_K H_2=0\end{gathered}$$

where

$$\begin{aligned}H_0&=L_0+M_0+G_0,\\ H_1&=L+G_1,\\ H_2&=L+G_3.\end{aligned}$$

Solving the above system of equations we get

$$\boldsymbol{\pi}_i=\boldsymbol{\pi}_{i-1}\mathcal{U}_i,\quad 1\le i\le K \tag{3}$$

where

$$\mathcal{U}_i=\begin{cases}-G[H_1+\mathcal{U}_2 M]^{-1}, & i=1,\\ -G_2[H_1+\mathcal{U}_{i+1} M]^{-1}, & 2\le i\le K-1\\ -G_2 H_2^{-1}, & i=K.\end{cases}$$

From the normalizing condition $\boldsymbol{\pi}\mathbf{e}=1$ we get

$$\boldsymbol{\pi}_0\left[I+\sum_{j=1}^{K}\prod_{i=1}^{j}\mathcal{U}_i\right]\mathbf{e}=1. \tag{4}$$

Theorem 1. *The queueing-inventory system under study is stable if and only if*

$$\boldsymbol{\pi}_0\mathcal{V}_0\boldsymbol{e}<\boldsymbol{\pi}_0\mathcal{V}_1\boldsymbol{e} \tag{5}$$

where

$$\begin{aligned}\mathcal{V}_0&=L_0+\sum_{j=1}^{K}\prod_{i=1}^{j}\mathcal{U}_i L\\ \mathcal{V}_1&=M_0+\mathcal{U}_1 M_1+\sum_{j=1}^{K}\prod_{i=2}^{j}\mathcal{U}_i M.\end{aligned}$$

Proof. The queueing-inventory system under study with the generator given in (1) is stable if and only if (see [32])

$$\boldsymbol{\pi} A_0\mathbf{e}<\boldsymbol{\pi} A_2\mathbf{e}. \tag{6}$$

Note that from the elements of A_0 and from A_2, we get

$$\boldsymbol{\pi} A_0\mathbf{e}=\boldsymbol{\pi}_0\left[L_0+\sum_{j=1}^{K}\prod_{i=1}^{j}\mathcal{U}_i L\right]\mathbf{e}\ \text{ and }\ \boldsymbol{\pi} A_2\mathbf{e}=\boldsymbol{\pi}_0\left[M_0+\mathcal{U}_1 M_1+\sum_{j=1}^{K}\prod_{i=2}^{j}\mathcal{U}_i M\right]\mathbf{e}.$$

Now using (6) we get the stated result. □

3. Steady state probability vector

Let $\mathbf{x}$ be the steady state probability vector of $\mathcal{Q}$. Then $\mathbf{x}$ must satisfy the set of equations

$$\mathbf{x}\mathcal{Q} = 0, \ \mathbf{x}\mathbf{e} = 1. \tag{7}$$

Thus the above set of equations reduce to:

$$\begin{aligned} \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} &= \mathbf{0}, \\ \mathbf{x}_0 A_{01} + \mathbf{x}_1 A_1 + \mathbf{x}_2 A_2 &= \mathbf{0}, \\ \mathbf{x}_{n-1} A_0 + \mathbf{x}_n A_1 + \mathbf{x}_{n+1} A_2 &= \mathbf{0}, \qquad n \geq 2. \end{aligned} \tag{8}$$

Under the assumption that the stability condition holds, we see that $\mathbf{x}$ is obtained as (see [32])

$$\mathbf{x}_n = \mathbf{x}_1 R^{n-1}, \qquad n \geq 2 \tag{9}$$

where R is the minimal non-negative solution to the matrix quadratic equation:

$$R^2 A_2 + R A_1 + A_0 = O \tag{10}$$

and the boundary equations are given by

$$\begin{aligned} \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} &= \mathbf{0}, \\ \mathbf{x}_0 A_{01} + \mathbf{x}_1 [A_1 + R A_2] &= \mathbf{0}. \end{aligned} \tag{11}$$

The normalizing condition (7) gives

$$\mathbf{x}_0 \left[I + \mathcal{M}(I - R)^{-1} \right] \mathbf{e} = 1 \tag{12}$$

where $\mathcal{M} = -A_{01}[A_1 + R A_2]^{-1}$.

3.1. Some important system performance measures

In this section, we list a few system performance measures along with their formulae, to bring out the qualitative nature of the model under study.

(1) Expected number of customers in the system:

$$E_N = \sum_{n=1}^{\infty} n \mathbf{x}_n \mathbf{e}.$$

(2) Expected number of customers in the orbit:

$$E_O = \sum_{n=0}^{\infty} \sum_{n_o=1}^{N} n_o \left[\sum_{i=0}^{S} \mathbf{x}_n(0, n_0, i) + \sum_{i_r=1}^{K} \mathbf{x}_n(i_r, n_0, S) \right] \mathbf{e}.$$

(3) Expected number of items in the main warehouse:

$$E_{\mathrm{SMW}} = \sum_{n=0}^{\infty} \left[\sum_{n_o=0}^{N} \sum_{i=1}^{S} i \mathbf{x}_n(0, n_0, i) + \sum_{i_r=1}^{K} \sum_{n_o=0}^{N} S \mathbf{x}_n(i_r, n_0, S) \right] \mathbf{e}.$$

(4) Expected number of items in the special warehouse:

$$E_{\mathrm{WRI}} = \sum_{n=0}^{\infty} \sum_{i_r=1}^{K} \sum_{n_o=0}^{N} i_r \mathbf{x}_n(i_r, n_0, S) \mathbf{e}.$$

(5) Expected rate of purchase:

$$E_{\mathrm{PR}} = \frac{1}{\mu'} \sum_{n=1}^{\infty} \sum_{n_o=0}^{N} \left[\sum_{i=1}^{S} \mathbf{x}_n(0, n_0, i) + \sum_{i_r=1}^{K} \mathbf{x}_n(i_r, n_0, S) \right] \mathbf{e}.$$

(6) Expected loss rate of customers due to no items in the system:

$$E_{\mathrm{LR}} = \phi_2 \lambda \sum_{n=0}^{\infty} \sum_{n_o=0}^{N} \mathbf{x}_n(0, n_0, 0) \mathbf{e}.$$

(7) Expected returned rate of items

$$E_{\mathrm{RI}} = \sigma_r \zeta_r \sum_{n=0}^{\infty} \sum_{n_o=0}^{N} \left[\sum_{i=0}^{S} \mathbf{x}_n(0, n_0, i) + \sum_{i_r=1}^{K-1} \mathbf{x}_n(i_r, n_0, S) \right] \mathbf{e}.$$

(8) Expected rate of customer feedback

$$E_{\mathrm{CF}} = \sigma_f \zeta_f \sum_{n=0}^{\infty} \sum_{n_o=0}^{N-1} \left[\sum_{i=0}^{S} \mathbf{x}_n(0, n_0, i) + \sum_{i_r=1}^{K} \mathbf{x}_n(i_r, n_0, S) \right] \mathbf{e}.$$

(9) Expected retrial rate of feedback customers

$$E_R = \eta \sum_{n_o=1}^{N} n_o \left[\sum_{i=1}^{S} \mathbf{x}_0(0, n_0, i) + \sum_{i_r=1}^{K} \mathbf{x}_0(i_r, n_0, S) \right] \mathbf{e}.$$

(10) Expected rate of replenishment

$$E_{\mathrm{RR}} = \theta \sum_{n=0}^{\infty} \sum_{n_o=0}^{N} \sum_{i=0}^{S-1} \mathbf{x}_n(0, n_0, i) \mathbf{e}.$$

4. Numerical examples

In order to bring out the qualitative nature of the system under study, we provide some illustrative examples in this section. We used the software MATLAB for solving the system numerically.

We assume that the maximum level of items in the special warehouse $K = 8$, the capacity of orbit is $N = 6$. PH service process of customer is characterized by

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -3 & 3 \\ 0 & -3 \end{pmatrix}$$

for which the mean service time $\mu' = 0.6667$.

For the arrival process, we consider the following two sets of values for D_0 and D_1 as follows.

(1) **MAP with negative correlation** (MAP$^-$)**:**

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225:75 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.9922 \\ 223.4925 & 0 & 2.2575 \end{pmatrix}.$$

TABLE 1. Effect of θ: fix $S = 6, s = 2, \eta = 3, \zeta_r = 3, \zeta_f = 2, \sigma_r = 0.3, \sigma_f = 0.2$.

θ	E_N	E_O	E_{SMW}	E_{WRI}	E_{PR}	E_{LR}	E_{RI}	E_{CF}	E_R	E_{RR}
	MAP$^+$									
0.5	25.5408	1.5279	4.7065	3.0464	1.0458	0.0031	0.8408	0.3642	0.3138	0.0737
1	25.1692	1.5232	4.8367	3.1845	1.0504	0.0010	0.8382	0.3654	0.3156	0.0992
1.5	25.1077	1.5224	4.8865	3.2433	1.0515	0.0005	0.8371	0.3657	0.3161	0.1147
2	25.1040	1.5223	4.9129	3.2774	1.0520	0.0003	0.8365	0.3658	0.3163	0.1252
	MAP$^-$									
0.5	2.3719	2.7008	5.1977	1.0006	1.3057	0.0042	0.8946	0.3338	0.3234	0.1842
1	2.3426	2.6573	5.5850	1.1838	1.3162	0.0006	0.8937	0.3384	0.3285	0.2497
1.5	2.3399	2.6510	5.7216	1.2713	1.3175	0.0001	0.8933	0.3389	0.3291	0.2883
2	2.3394	2.6496	5.7907	1.3249	1.3178	0.0000	0.8931	0.3390	0.3292	0.3142

TABLE 2. Effect of η: fix $S = 6, s = 2, \theta = 2, \zeta_r = 3, \zeta_f = 2, \sigma_r = 0.3, \sigma_f = 0.2$.

η	E_N	E_O	E_{SMW}	E_{WRI}	E_{PR}	E_{LR}	E_{RI}	E_{CF}	E_R	E_{RR}
	MAP$^+$									
1.5	29.9445	1.9151	4.9056	3.2016	1.0778	0.0007	0.8398	0.3578	0.2934	0.1337
2	28.4406	1.7523	4.9076	3.2239	1.0701	0.0004	0.8387	0.3605	0.3009	0.1314
2.5	26.7941	1.6274	4.9102	3.2500	1.0613	0.0003	0.8376	0.3631	0.3086	0.1284
3	25.1040	1.5223	4.9129	3.2774	1.0520	0.0001	0.8365	0.3658	0.3163	0.1252
	MAP$^-$									
1.5	2.3201	3.0586	5.7968	1.3333	1.3055	0.0001	0.8932	0.3270	0.3168	0.3073
2	2.3298	2.8637	5.7938	1.3271	1.3116	0.0000	0.8932	0.3330	0.3230	0.3109
2.5	2.3356	2.7376	5.7919	1.3252	1.3153	0.0000	0.8931	0.3366	0.3267	0.3129
3	2.3394	2.6496	5.7907	1.3249	1.3178	0.0000	0.8931	0.3390	0.3292	0.3142

(2) **MAP with positive correlation** (MAP$^+$)**:**

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225:75 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.9922 & 0 & 0.01002 \\ 2.2575 & 0 & 223.4925 \end{pmatrix}.$$

The above MAP processes will be normalized so as to have a specific arrival rate. However, these are qualitatively different in that they have different correlation structure. The arrival processes labeled MAP$^-$ has negative correlation with value -0.4889 and MAP$^+$ has positive correlation with value 0.4889 for two successive inter-arrival times.

Tables 1–4 provide the effect of values of replenishment rate (θ), retrial rate (η), returned rate of items (ζ_r) and rate of customer feedback (ζ_f), the effect of MAP with positive and negative correlation on the expected number of customers in the system and the orbit, expected number of items in the main warehouse and special warehouse, expected returned rate of items, expected purchase rate, loss rate, rate of customer feedback, retrial rate and rate of replenishment are indicated.

A quick look at Tables 1–4 reveal some interesting observations.

– As θ is increased, as expected, E_{LR} decreases for both arrival processes. However, the rate of decrease is much higher for MAP$^-$ but MAP$^+$ indicating the role of (positive) correlated arrivals.
– While the expected inventory level in the main warehouse and special warehouse increase as θ increases, as is to be expected, the rate of increase is pretty much the same for both arrival processes.

TABLE 3. Effect of ζ_r: fix $S=6, s=2, \theta=2, \eta=3, \zeta_f=2, \sigma_r=0.3, \sigma_f=0.2$.

ζ_r	E_N	E_O	E_{SMW}	E_{WRI}	E_{PR}	E_{LR}	E_{RI}	E_{CF}	E_R	E_{RR}
	MAP$^+$									
1.5	25.1100	1.5225	4.7401	0.7938	1.0516	0.0005	0.4486	0.3657	0.3161	0.3919
2	25.1057	1.5224	4.8078	1.3367	1.0518	0.0004	0.5951	0.3657	0.3162	0.2893
2.5	25.1038	1.5223	4.8679	2.2044	1.0519	0.0004	0.7295	0.3657	0.3162	0.1960
3	25.1040	1.5223	4.9129	3.2774	1.0520	0.0003	0.8365	0.3658	0.3163	0.1252
	MAP$^-$									
1.5	2.3400	2.6512	5.5710	0.3089	1.3175	0.0001	0.4498	0.3389	0.3291	0.6130
2	2.3397	2.6505	5.6454	0.5134	1.3176	0.0001	0.5995	0.3390	0.3292	0.5157
2.5	2.3395	2.6500	5.7192	0.8324	1.3177	0.0001	0.7482	0.3390	0.3292	0.4152
3	2.3394	2.6496	5.7907	1.3249	1.3178	0.0000	0.8931	0.3390	0.3292	0.3142

TABLE 4. Effect of ζ_f: fix $S=6, s=2, \theta=2, \zeta_r=3, \eta=3, \sigma_r=0.3, \sigma_f=0.2$.

ζ_f	E_N	E_O	E_{SMW}	E_{WRI}	E_{PR}	E_{LR}	E_{RI}	E_{CF}	E_R	E_{RR}
	MAP$^+$									
0.5	24.8463	0.8415	4.9463	5.2082	0.8265	0.0002	0.6815	0.0957	0.0903	0.0702
1	24.9328	1.0267	4.9410	4.6314	0.9024	0.0001	0.7456	0.1887	0.1664	0.0791
1.5	25.0190	1.2492	4.9302	3.9571	0.9778	0.0001	0.7983	0.2790	0.2419	0.0969
2	25.1040	1.5223	4.9129	3.2774	1.0520	0.0001	0.8365	0.3658	0.3163	0.1252
	MAP$^-$									
0.5	1.9734	0.4969	5.8891	2.4555	1.0931	0.0000	0.8665	0.0997	0.1025	0.1706
1	2.1168	1.1431	5.8523	1.9413	1.1815	0.0000	0.8809	0.1950	0.1917	0.2252
1.5	2.2418	1.9040	5.8182	1.5698	1.2581	0.0000	0.8889	0.2765	0.2690	0.2748
2	2.3394	2.6496	5.7907	1.3249	1.3178	0.0000	0.8931	0.3390	0.3292	0.3142

– As η is increased, E_{SMW} and E_{WRI} increase for MAP$^+$. However, both rates decrease for MAP$^-$.
– We notice that the measures, E_{PR} and E_{RR}, decrease as η increases for positive correlation. However, these rates increase for MAP$^-$ when η increases.
– As ζ_r is increased, E_{WRI} increases for both arrival processes. However, the rate of increase is much higher for MAP$^+$. Also, E_{RI} increases for both arrival processes in similar way.
– Table 4 shows that $E_N, E_O, E\text{PR}, E_{\text{RI}}, E_{\text{CF}}, E_R, E_{\text{RR}}$ increase as ζ_f increases for both arrivals, as expected line. However, $E_{SWM}, E_{\text{WRI}}, E_{\text{LR}}$ decrease.

4.1. Revenue (profit) function

Based on performance measures we define the following revenue (profit) function as:
$F(\sigma_r, \sigma_f, S, K, N) = C_1E_{\text{PR}} + C_2E_R + C_3E_{\text{CF}} - (C_4E_{\text{RI}} + C_5E_{\text{SMW}} + C_6E_{\text{WRI}} + C_7E_N + C_8E_O + C_9E_{\text{LR}} + [\mathbf{K} + C_{10}(S-s)]E_{\text{RR}})$ where

– C_1 = revenue to the system due to per unit purchase (by a customer at the end of his service)
– C_2 = revenue due to successful retrial of feedback customer per unit time
– C_3 = revenue due to feedback of customer
– C_4 = cost to the system from each return of purchased item
– C_5 = holding cost per inventoried item in main warehouse per unit time
– C_6 = holding cost per inventoried item in special warehouse per unit time
– C_7 = holding cost per customer in the infinite queue

TABLE 5. Effect of s and S.

S	MAP$^+$	MAP$^-$
	$s = 1$	
2	45.8330	94.2452
3	43.1744	91.1515
4	40.5941	88.8721
5	37.9698	86.5631
6	35.3344	84.2474
	$s = 2$	
3	43.3744	91.4108
4	40.8271	89.1366
5	38.2316	86.8319
6	35.6214	84.5198
	$s = 3$	
4	41.0600	89.4011
5	38.4934	87.1006
6	35.9084	84.7922

TABLE 6. Optimum value of K and corresponding value of revenue.

K	MAP$^+$			K	MAP$^-$		
	$N = 1$	$N = 2$	$N = 3$		$N = 1$	$N = 2$	$N = 3$
2	30.0816	33.9563	35.7259	2	70.7476	74.1139	75.4695
3	**32.6439**	37.1579	39.1074	3	76.3020	80.0856	81.6224
4	32.5564	**37.5180**	**39.6215**	4	78.4654	82.6679	84.3991
5	31.3932	36.6251	38.8383	5	**78.8423**	**83.4557**	**85.3936**
6	29.7544	35.1338	37.4125	6	78.1595	83.1668	85.3176
7	27.8841	33.3372	35.6474	7	76.8023	82.1797	84.5436
8	25.8886	31.3743	33.6934	8	74.9977	80.7172	83.2897
9	23.8170	29.3135	31.6280	9	72.8890	78.9203	81.6934

– C_8= holding cost per customer in the finite orbit
– C_9 = cost due to customer lost per unit time
– **K** = fixed cost of delivery service
– C_{10} = carriage cost of the delivery service per item.

In order to study the effect of different parameters on profit function we first take the values

$$(C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, \mathbf{K}, C_{10}) = (\$100, \$25, \$5, \$10, \$2, \$3, \$1, \$2, \$15, \$110, \$8).$$

We assign the following values to the parameters: $K = 9, N = 6, \phi_1 = 0.6, \theta = 2, \eta = 3, \zeta_r = 3, \zeta_f = 2, \sigma_r = 0.5, \sigma_f = 0.2$. For different values of S and s, the expected profit is calculated and presented in Table 5. This table shows that the profit function decreases when S increases and increases for s.

Table 6 represents the revenue to the system as a function of K and N for MAP with positive and negative correlations (fix $s = 2, S = 6$). Whereas MAP with positive correlation provides lower revenue, MAP with negative correlation gives much higher revenue. In both cases the cost function behaves as a concave function (see Figs. 2 and 3).

From Table 7 we obtain as σ_r is increased, as expected, revenue increases then decreases for both arrival processes (fix $S = 7, s = 2, N = 6, K = 9$). However, the rate of increase is much higher for MAP$^+$ (see Fig. 4).

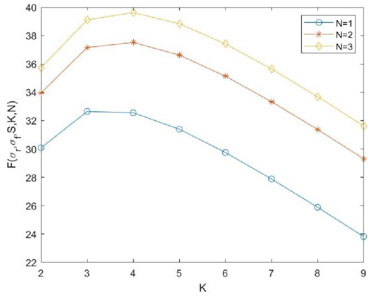

FIGURE 2. Effect of K, N, MAP^{+}.

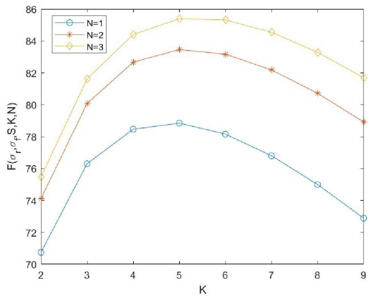

FIGURE 3. Effect of K, N, MAP^{-}.

TABLE 7. Effect of σ_r.

σ_r	MAP$^+$	MAP$^-$
0	-40.2524	-19.3677
0.1	-11.4736	4.8640
0.2	-1.4760	30.1947
0.3	-0.3610	48.8050
0.4	0.1320	**53.0792**
0.5	**0.4093**	51.7235
0.6	0.3962	50.3904
0.7	0.2174	49.5696
0.8	-0.0216	49.0581
0.9	-0.2612	48.7173
1	-0.4754	48.4759

TABLE 8. Effect of $\eta, \theta, \zeta_r, \zeta_f$ on $F(\sigma_r, \sigma_f, S, K, N)$.

$\theta=2, \zeta_r=3, \zeta_f=2$			$\eta=3, \zeta_r=3, \zeta_f=2$			$\theta=2, \eta=3, \zeta_f=2$			$\theta=2, \zeta_r=3, \eta=3$		
η	MAP$^+$	MAP$^-$	θ	MAP$^+$	MAP$^-$	ζ_r	MAP$^+$	MAP$^-$	ζ_f	MAP$^+$	MAP$^-$
1	31.9839	82.3915	1	40.1856	87.4062	1	13.4894	30.5611	1	22.5403	74.5599
2	35.6240	85.0651	2	38.9844	86.0034	2	37.9831	68.1502	2	38.9844	86.0034
3	38.9844	86.0034	3	38.3361	85.2894	3	38.9844	86.0034	3	53.8385	90.3595

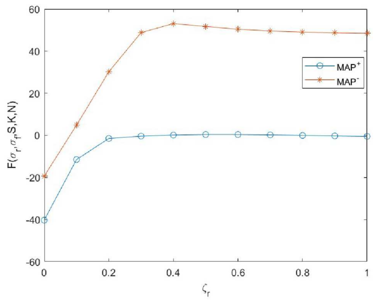

FIGURE 4. Effect of σ_r when $\sigma_f = 0$.

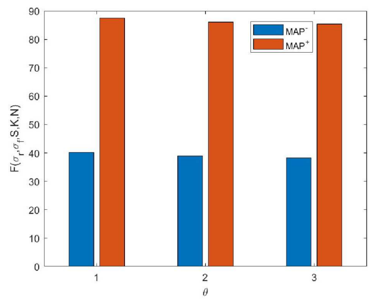

FIGURE 5. Effect of θ.

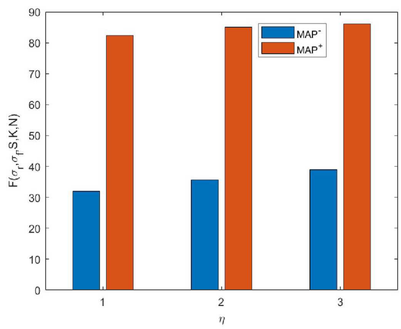

FIGURE 6. Effect of η.

The effect of retrial, replenishment, return of item and customer feedback parameters on revenue function are given in Table 8 (see Figs. 5–8) for MAP with positive and negative correlations. With increase in values of η, ζ_r, ζ_f, revenue increase in both cases; however, the effect of replenishment rate θ is increasing the revenue decreases. These behaviours are on expected lines.

5. SPECIAL CASE

Next we consider a special case of the model described in Section 3. In this section customers arrive according to a Poisson process of rate λ. To deliver one unit of the item to the customer in service, it requires an

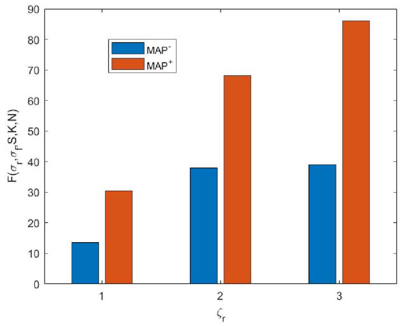

FIGURE 7. Effect of ζ_r.

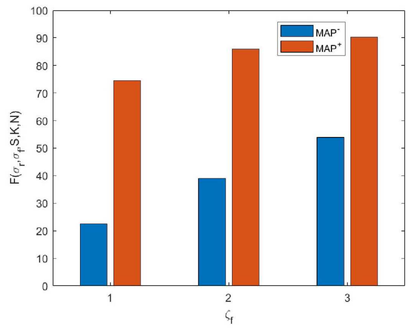

FIGURE 8. Effect of ζ_f.

exponentially distributed amount of time with parameter μ. Other assumptions remain the same as described above in Section 3.

Let $N_S(t)$, $N_O(t)$, $I_r(t)$, and $I(t)$ denote, respectively, the number of customers in the system including the one in service, the number of feedback customers in the orbit, the number of items in the special warehouse (WRI) and the number of items in the main warehouse at time t. The process $\tilde{\Omega} = \{(N_S(t), I_r(t), N_O(t), I(t)), t \geq 0\}$ is a CTMC with the state space given by

$$\{(n, 0, n_o, i), n \geq 0, 0 \leq n_o \leq N, 0 \leq i \leq S\} \bigcup \{(n, i_r, n_o, S), n \geq 0, 1 \leq i_r \leq K, 0 \leq n_o \leq N\}.$$

The transition rates are:

(i) Transitions due to the arrival:

$$\begin{array}{lll} (n, 0, n_0, 0) \to (n+1, 0, n_0, 0): & \text{rate } \ \phi_1\lambda & \text{for } n \geq 0, 0 \leq n_o \leq N \\ (n, 0, n_0, i) \to (n+1, 0, n_0, i): & \text{rate } \ \lambda & \text{for } n \geq 0, 0 \leq n_o \leq N, 1 \leq i \leq S \\ (n, i_r, n_0, S) \to (n+1, i_r, n_0, S): & \text{rate } \ \lambda & \text{for } n \geq 0, 1 \leq i_r \leq K, 0 \leq n_o \leq N. \end{array}$$

(ii) Transitions due to retrial:

$$\begin{array}{lll} (0, 0, n_o, i) \to (1, 0, n_o - 1, i): & \text{rate } \ n_o\eta & \text{for } 1 \leq n_o \leq N, 1 \leq i \leq S \\ (0, i_r, n_0, S) \to (1, i_r, n_0 - 1, S): & \text{rate } \ n_o\eta & \text{for } 1 \leq i_r \leq K, 1 \leq n_o \leq N. \end{array}$$

(iii) Transitions due to service completion:

$$\begin{array}{lll} (n, 0, n_o, i) \to (n-1, 0, n_o, i-1): & \text{rate } \ \mu & \text{for } n \geq 1, 0 \leq n_o \leq N, 1 \leq i \leq S \\ (n, i_r, n_0, S) \to (n-1, i_r - 1, n_0, S): & \text{rate } \ \mu & \text{for } , n \geq 1, 1 \leq i_r \leq K, 0 \leq n_o \leq N. \end{array}$$

(iv) Transitions due to replenishment:

$$(n,0,n_o,i) \to (n,0,n_o,S): \text{ rate } \theta \text{ for } n \geq 0, 0 \leq n_o \leq N, 0 \leq i \leq S-1.$$

(v) Transitions due to return of items:

$$\begin{array}{lll} (n,0,n_o,i) \to (n,0,n_o,i+1): & \text{rate } \sigma_r\zeta_r & \text{for } n \geq 0, 0 \leq n_o \leq N, 0 \leq i \leq S-1 \\ (n,i_r,n_o,S) \to (n,i_r+1,n_o,S): & \text{rate } \sigma_r\zeta_r & \text{for } n \geq 0, 0 \leq i_r \leq K-1, 0 \leq n_o \leq N. \end{array}$$

(vi) Transitions due to feedback of customers:

$$\begin{array}{lll} (n,0,n_o,i) \to (n,0,n_o+1,i): & \text{rate } \sigma_f\zeta_f & \text{for } n \geq 0, 0 \leq n_o \leq N-1, 0 \leq i \leq S \\ (n,i_r,n_o,S) \to (n,i_r,n_o+1,S): & \text{rate } \sigma_f\zeta_f & \text{for } n \geq 0, 1 \leq i_r \leq K, 0 \leq n_o \leq N-1. \end{array}$$

Thus the infinitesimal generator of $\tilde{\Omega}$ is of the form

$$\tilde{\mathcal{Q}} = \begin{pmatrix} \tilde{A}_{00} & \tilde{A}_{01} & & & \\ \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & & \\ & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \\ & \ddots & \ddots & & \ddots \end{pmatrix}. \tag{13}$$

Each matrix $\tilde{A}_{00}, \tilde{A}_{01}, \tilde{A}_0, \tilde{A}_1, \tilde{A}_2$ is square matrix of order $(N+1)(S+K+1)$ where

$$\tilde{A}_0 = \begin{pmatrix} \mathcal{L}_0 & & & \\ & \lambda I & & \\ & & \ddots & \\ & & & \lambda I \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} \mathcal{M}_0 & & & & \\ \mathcal{M}_1 & & & & \\ & \mu I & & & \\ & & \ddots & & \\ & & & \mu I & \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} \mathcal{G}_{00} & \mathcal{G}_0 & & & \\ & \mathcal{G}_1 & \sigma_r\zeta_r I & & \\ & & \ddots & \ddots & \\ & & & \mathcal{G}_1 & \sigma_r\zeta_r I \\ & & & & \mathcal{G}_1' \end{pmatrix},$$

$$\tilde{A}_{01} = \begin{pmatrix} \hat{\mathcal{L}}_0 & & & \\ & \hat{\mathcal{L}} & & \\ & & \ddots & \\ & & & \hat{\mathcal{L}} \end{pmatrix}, \quad \tilde{A}_{00} = \begin{pmatrix} \hat{\mathcal{G}}_{00} & \mathcal{G}_0 & & & \\ & \hat{\mathcal{G}}_1 & \sigma_r\zeta_r I & & \\ & & \ddots & \ddots & \\ & & & \hat{\mathcal{G}}_1 & \sigma_r\zeta_r I \\ & & & & \hat{\mathcal{G}}_1' \end{pmatrix}$$

and all other sub-matrices are given in Appendix B.

5.1. Stability condition

Let $\tilde{\boldsymbol{\pi}}$ be the steady state probability vector of

$$\tilde{A} = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 = \begin{pmatrix} \mathcal{H}_0 & \mathcal{G}_0 & & & & \\ \mathcal{M}_1 & \mathcal{H}_1 & \sigma_r\zeta_r I & & & \\ & \mu I & \mathcal{H}_1 & \sigma_r\zeta_r I & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu I & \mathcal{H}_1 & \sigma_r\zeta_r I \\ & & & & \mu I & \mathcal{H}_2 \end{pmatrix}.$$

Then

$$\tilde{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}, \ \tilde{\boldsymbol{\pi}}\mathbf{e} = 1. \tag{14}$$

From (14) we have

$$\begin{aligned}
\tilde{\boldsymbol{\pi}}_0\mathcal{H}_0 + \tilde{\boldsymbol{\pi}}_1\mathcal{M}_1 &= 0,\\
\tilde{\boldsymbol{\pi}}_0\mathcal{G}_0 + \tilde{\boldsymbol{\pi}}_1\mathcal{H}_1 + \mu\tilde{\boldsymbol{\pi}}_2 I &= 0,\\
\sigma_r\zeta_r\tilde{\boldsymbol{\pi}}_{i-1}I + \tilde{\boldsymbol{\pi}}_i\mathcal{H}_1 + \mu\tilde{\boldsymbol{\pi}}_{i+1}I &= 0, \qquad 2 \le i \le K-1\\
\sigma_r\zeta_r\tilde{\boldsymbol{\pi}}_{K-1}I + \tilde{\boldsymbol{\pi}}_K\mathcal{H}_2 &= 0
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_0 =&\mathcal{L}_0 + \mathcal{G}_{00} + \mathcal{M}_0,\\
\mathcal{H}_1 =&\lambda I + \mathcal{G}_1,\\
\mathcal{H}_2 =&\lambda I + \mathcal{G}_1'.
\end{aligned}$$

Solving the above system of equations (see Appendix C) we get

$$\tilde{\boldsymbol{\pi}}_i = \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{U}}_i, \qquad 0 \le i \le K-1 \tag{15}$$

where

$$\tilde{\mathcal{U}}_i = \begin{cases} -\left[\tilde{\mathcal{U}}_1\mathcal{M}_1\right]\mathcal{H}_0^{-1}, & i = 0,\\ -\left[\frac{\tilde{\mathcal{U}}_{i+1}\mathcal{H}_1 + \mu\tilde{\mathcal{U}}_{i+2}}{\sigma_r\zeta_r}\right], & 1 \le i \le K-3\\ -\left[\frac{\tilde{\mathcal{U}}_{K-1}\mathcal{H}_1 + \mu I}{\sigma_r\zeta_r}\right], & i = K-2,\\ -\frac{\mathcal{H}_2}{\sigma_r\zeta_r}, & i = K-1. \end{cases}$$

From the normalizing condition $\tilde{\boldsymbol{\pi}}\mathbf{e} = 1$ we get

$$\tilde{\boldsymbol{\pi}}_K\left[I + \sum_{j=0}^{K-1}\tilde{\mathcal{U}}_j\right]\mathbf{e} = 1. \tag{16}$$

Theorem 2. *The queueing-inventory system under study is stable if and only if*

$$\tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{V}}_0\mathbf{e} < \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{V}}_1\mathbf{e} \tag{17}$$

where

$$\begin{aligned}
\tilde{\mathcal{V}}_0 &= \tilde{\mathcal{U}}_0\mathcal{L}_0 + \lambda\left(\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right)\\
\tilde{\mathcal{V}}_1 &= \tilde{\mathcal{U}}_0\mathcal{M}_0 + \mu\left(\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right).
\end{aligned}$$

Proof. The queueing-inventory system under study with the generator given in (13) is stable if and only if (see [32])

$$\tilde{\boldsymbol{\pi}}\tilde{A}_0\mathbf{e} < \tilde{\boldsymbol{\pi}}\tilde{A}_2\mathbf{e}. \tag{18}$$

Note that from the elements of $\tilde{A}_0$ and from $\tilde{A}_2$, we get

$$\tilde{\boldsymbol{\pi}}\tilde{A}_0\mathbf{e} = \tilde{\boldsymbol{\pi}}_0\mathcal{L}_0\mathbf{e} + \lambda\tilde{\boldsymbol{\pi}}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}$$

and

$$\tilde{\boldsymbol{\pi}}\tilde{A}_2\mathbf{e} = \tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} + \tilde{\boldsymbol{\pi}}_1\mathcal{M}_1\mathbf{e} + \mu\boldsymbol{\pi}_K\left[\sum_{j=2}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}$$
$$= \tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} + \mu\boldsymbol{\pi}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}.$$

Now using (18) we get the stated result. □

Note:
We have $\tilde{\boldsymbol{\pi}}_0\mathcal{L}_0\mathbf{e} = \phi_1\lambda\sum_{n_0=0}^{N}\tilde{\pi}(0,n_0,0) + \lambda\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i)$ and $\tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} = \mu\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i)$. Hence the queueing-inventory system under study with $\phi_1 = 0$ is stable if and only if $\lambda < \mu$ (see Appendix C). This fact is very important: in the Poisson/exponential model, in the case of a lost sale scheme, the ergodicity condition does not depend either on the capacity of warehouses, or on the lead time, or on the intensity of repeated and feedback customers, or on the intensity of the return of purchased items and coincides with the well-known ergodicity condition of a classical single-line queuing system. Same result was established for similar models, see [14, 26, 33].

5.2. Steady state probability vector

Let $\tilde{\mathbf{x}}$ be the steady state probability vector of $\tilde{\mathcal{Q}}$. Then $\tilde{\mathbf{x}}$ must satisfy the set of equations

$$\tilde{\mathbf{x}}\tilde{\mathcal{Q}} = 0, \tilde{\mathbf{x}}\mathbf{e} = 1. \tag{19}$$

Thus the above set of equations reduce to:

$$\begin{aligned}\tilde{\mathbf{x}}_0\tilde{A}_{00} + \tilde{\mathbf{x}}_1\tilde{A}_2 &= \mathbf{0},\\ \tilde{\mathbf{x}}_0\tilde{A}_{01} + \tilde{\mathbf{x}}_1\tilde{A}_1 + \tilde{\mathbf{x}}_2\tilde{A}_2 &= \mathbf{0},\\ \tilde{\mathbf{x}}_{n-1}\tilde{A}_0 + \tilde{\mathbf{x}}_n\tilde{A}_1 + \tilde{\mathbf{x}}_{n+1}\tilde{A}_2 &= \mathbf{0}, \qquad n \geq 2.\end{aligned} \tag{20}$$

Under the assumption that the stability condition (18) holds, we see that $\mathbf{x}$ is obtained as (see [32])

$$\tilde{\mathbf{x}}_n = \tilde{\mathbf{x}}_1\tilde{R}^{n-1}, \qquad n \geq 2 \tag{21}$$

where $\tilde{R}$ is the minimal non-negative solution to the matrix quadratic equation:

$$\tilde{R}^2\tilde{A}_2 + \tilde{R}\tilde{A}_1 + \tilde{A}_0 = O \tag{22}$$

and the boundary equations are given by

$$\begin{aligned}\tilde{\mathbf{x}}_0\tilde{A}_{00} + \tilde{\mathbf{x}}_1\tilde{A}_2 &= \mathbf{0},\\ \tilde{\mathbf{x}}_0\tilde{A}_{01} + \tilde{\mathbf{x}}_1\left[\tilde{A}_1 + \tilde{R}\tilde{A}_2\right] &= \mathbf{0}.\end{aligned} \tag{23}$$

The normalizing condition (19) gives

$$\tilde{\mathbf{x}}_0\left[I + \tilde{\mathcal{M}}(I-\tilde{R})^{-1}\right]\mathbf{e} = 1 \tag{24}$$

where $\tilde{\mathcal{M}} = -\tilde{A}_{01}[\tilde{A}_1 + \tilde{R}\tilde{A}_2]^{-1}$.

5.3. Performance measures

(1) Expected number of customers in the system:

$$\tilde{E}_N = \sum_{n=1}^{\infty} n\tilde{\mathbf{x}}_n\mathbf{e}.$$

(2) Expected number of customers in the orbit:

$$\tilde{E}_O = \sum_{n=0}^{\infty}\sum_{n_o=1}^{N} n_o\left[\sum_{i=0}^{S}\tilde{x}_n(0,n_0,i) + \sum_{i_r=1}^{K}\tilde{x}_n(i_r,n_0,S)\right].$$

(3) Expected number of items in the main warehouse:

$$\tilde{E}_{\mathrm{SMW}} = \sum_{n=0}^{\infty}\left[\sum_{n_o=0}^{N}\sum_{i=1}^{S} i\tilde{x}_n(0,n_0,i) + \sum_{i_r=1}^{K}\sum_{n_o=0}^{N} S\tilde{x}_n(i_r,n_0,S)\right].$$

(4) Expected number of items in the special warehouse:

$$\tilde{E}_{\mathrm{WRI}} = \sum_{n=0}^{\infty}\sum_{i_r=1}^{K}\sum_{n_o=0}^{N} i_r\tilde{x}_n(i_r,n_0,S).$$

(5) Expected rate of purchase:

$$\tilde{E}_{\mathrm{PR}} = \mu\sum_{n=1}^{\infty}\sum_{n_o=0}^{N}\left[\sum_{i=1}^{S}\tilde{x}_n(0,n_0,i) + \sum_{i_r=1}^{K}\tilde{x}_n(i_r,n_0,S)\right].$$

(6) Expected loss rate of customers due to no items in the system:

$$\tilde{E}_{\mathrm{LR}} = \phi_2\lambda\sum_{n=0}^{\infty}\sum_{n_o=0}^{N}\tilde{x}_n(0,n_0,0).$$

(7) Expected returned rate of items

$$\tilde{E}_{\mathrm{RI}} = \sigma_r\zeta_r\sum_{n=0}^{\infty}\sum_{n_o=0}^{N}\left[\sum_{i=0}^{S}\tilde{x}_n(0,n_0,i) + \sum_{i_r=1}^{K-1}\tilde{x}_n(i_r,n_0,S)\right].$$

(8) Expected rate of customer feedback

$$\tilde{E}_{\mathrm{CF}} = \sigma_f\zeta_f\sum_{n=0}^{\infty}\sum_{n_o=0}^{N-1}\left[\sum_{i=0}^{S}\tilde{x}_n(0,n_0,i) + \sum_{i_r=1}^{K}\tilde{x}_n(i_r,n_0,S)\right].$$

(9) Expected retrial rate of feedback customers

$$\tilde{E}_R = \eta\sum_{n_o=1}^{N} n_o\left[\sum_{i=1}^{S}\tilde{x}_0(0,n_0,i) + \sum_{i_r=1}^{K}\tilde{x}_0(i_r,n_0,S)\right].$$

(10) Expected rate of replenishment

$$\tilde{E}_{\mathrm{RR}} = \theta\sum_{n=0}^{\infty}\sum_{n_o=0}^{N}\sum_{i=0}^{S-1}\tilde{x}_n(0,n_0,i).$$

TABLE 9. Optimum value of K and corresponding value of revenue.

K	$N=1$	$N=2$	$N=3$
34	91.8035	94.8375	96.3941
35	91.8346	94.9212	96.5041
36	**91.8478**	94.9876	96.5974
37	91.8431	95.0370	96.6739
38	91.8206	95.0694	96.7337
39	91.7806	**95.0849**	96.7769
40	91.7231	95.0836	96.8036
41	91.6484	95.0657	**96.8140**
42	91.5565	95.0313	96.8081
43	91.4478	94.9805	96.7860
44	91.3222	94.9134	96.7479

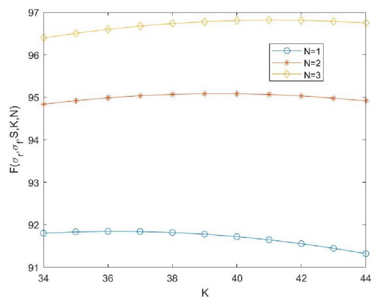

FIGURE 9. Effect of K, N.

5.4. Revenue function

Here we define the revenue function as $F(\sigma_r, \sigma_f, S, K, N) = C_1\tilde{E}_{\mathrm{PR}} + C_2\tilde{E}_R + C_3\tilde{E}_{\mathrm{CF}} - (C_4\tilde{E}_{\mathrm{RI}} + C_5\tilde{E}_{\mathrm{SMW}} + C_6\tilde{E}_{\mathrm{WRI}} + C_7\tilde{E}_N + C_8\tilde{E}_O + C_9\tilde{E}_{\mathrm{LR}} + [\mathbf{K} + C_{10}(S-s)]\tilde{E}_{\mathrm{RR}})$ where C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, $\mathbf{K}$, C_{10} are given in Section 4.1.

Fix $(C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, \mathbf{K}, C_{10}) = (\$100, \$25, \$5, \$10, \$2, \$3, \$1, \$2, \$15, \$110, \$8)$. Also assign the following values to the parameters: $S = 18, s = 7, \phi_1 = 0.6, \lambda = 1, \mu = 1.5, \theta = 2, \eta = 3, \zeta_r = 3, \zeta_f = 2, \sigma_r = 0.5, \sigma_f = 0.2$. For different values of K and N, the expected profit is calculated and presented in Table 9. This table shows that the profit function behaves as a concave function (see Fig. 9).

Take $S = 15, s = 7, K = 25, N = 18, \phi_1 = 0.6, \lambda = 1, \mu = 1.5, \theta = 2, \eta = 3, \zeta_r = 3, \zeta_f = 2$. Table 10 (see Fig. 10) gives the effect of σ_r when $\sigma_f = 0$ and σ_f when $\sigma_r = 0$. In both cases the revenue increased initially with the increased values of σ_r (or σ_f), it later showed a decreasing behaviour (at $\sigma_r = 0.6$ (or $\sigma_f = 0.6$)).

6. CONCLUSION

In this paper, we propose an infinite single-server QIS model with return of purchased items sold from unsatisfied primary customers and feedback from already served satisfied customers for the purchase of a new batch of items. Primary customers form a MAP flow and their service times has PH-distribution. Unlike the classical models of QISs, here it is assumed that the system has two warehouses: system main warehouse (SMW) and warehouse for returned items (WRI). Both warehouses have finite volumes, and it is assumed that the cost of holding one inventory item in different warehouses varies. Unlike primary customers, feedback customers are generated by returning items and form a virtual orbit of finite size, with their sojourn time in orbit distributed

TABLE 10. Optimum value of σ_r, σ_f and corresponding value of revenue.

$\sigma_f = 0$		$\sigma_r = 0$	
σ_r	$F(\sigma_r, \sigma_f, S, K, N)$	σ_f	$F(\sigma_r, \sigma_f, S, K, N)$
0	11.4255	0	11.4255
0.1	27.8028	0.1	22.8157
0.2	45.8604	0.2	33.5876
0.3	65.3894	0.3	42.8973
0.4	85.7771	0.4	49.6515
0.5	104.9475	0.5	53.1963
0.6	**115.8214**	0.6	**54.1778**
0.7	111.4602	0.7	53.9083
0.8	103.5883	0.8	53.2899
0.9	98.4366	0.9	52.6848
1	95.1014	1	52.1847

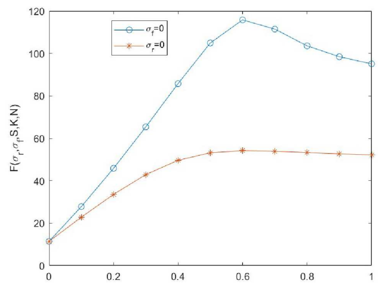

FIGURE 10. Effect of σ_r, σ_f.

exponentially. The returned item is considered new and it goes directly to the SMW if at least one free space there; otherwise, this item is sent to WRI. After completing service of each customer, one item is instantly sent to the SMW from the WRI (if any). In SMW, for regular ordering known "Up to S replenishment policy is used with positive lead time that has exponential distribution. When the stock level reaches the maximum value due to returning of items, the system immediately cancels the regular order. A combination of lost sales and backorder sales schemes is used, *i.e.* a new customer joins the queue even at zero inventory level with a positive probability.

The mathematical model of the studied QIS is formulated as a multi-dimensional Markov chain with an infinite state space. The stability condition for the constructed Markov chain is obtained. For the Poisson/exponential model, it is shown that in the case of a lost sale scheme, the stability condition depends only on the intensity of primary customers and their service time and does not depend on other parameters of the system. Under stability conditions, system performance measures are calculated *via* steady-state probabilities, which are determined using the matrix-geometric method. Unlike classic QISs, several new performance measures are defined here, such as the expected level of inventory in the main and special warehouses, the expected rate of items returns, the expected rate of repeated feedback customers, etc.

Using numerical examples, the behavior of performance measures is studied and analyzed depending on the both load and structural parameters of the studied QIS. The maximization of the revenue function is also performed numerically for both cases of MAP with positive and negative correlations.

As a direction for further research, we can indicate a generalization of the obtained results for models of perishable QIS, as well as a study of a similar model with an infinite volume of WRI (for simplicity this paper assumes that returns are not allowed if the WRI is full).

Acknowledgments

We would like to thank the anonymous reviewers for their valuable and insightful comments that helped us significantly improve the paper.

Data availability statement

No new data/codes were created or analyzed in this study.

References

[1] A. Aghsami, Y. Samimi and A. Aghaei, A novel Markovian queueing-inventory model with imperfect production and inspection processes: a hospital case study. *Comput. Ind. Eng.* **162** (2021) 107772.

[2] A. Aghsami, Y. Samimi and A. Aghaei, An integrated Markovian queueing-inventory model in a single retailer-single supplier problem with imperfect quality and destructive testing acceptance sampling. *Adv. Ind. Eng.* **55** (2021) 367–401.

[3] M. Amirthakodi and B. Sivakumar, An inventory system with service facility and finite orbit size for feedback customers. *OPSEARCH* **52** (2015) 225–255.

[4] M. Amirthakodi and B. Sivakumar, An inventory system with service facility and feedback customers. *Int. J. Ind. Syst. Eng.* **33** (2019) 374–411.

[5] M. Amirthakodi, V. Radhamami and B. Sivakumar, A perishable inventory system with service facility and feedback customers. *Ann. Oper. Res.* **233** (2015) 25–55.

[6] G. Ayyappan and S. Karpagam, Analysis of a bulk queue with unreliable server, immediate feedback, N-policy, Bernoulli schedule multiple vacation and stand-by server. *Ain Shams Eng. J.* **10** (2019) 873–880.

[7] G. Ayyappan and K. Thilagavathy, Analysis of $MAP/PH/1$ queueing model with immediate feedback, starting failures, single vacation, standby server, delayed repair, breakdown and impatient customers. *Int. J. Math. Oper. Res.* **19** (2021) 269–301.

[8] J.W. Baek, On the control policy of a queuing? Inventory system with variable inventory replenishment speed. *Mathematics* **12** (2024) 194.

[9] A.A. Bouchentouf, C. Mouloud and M. Boualem, Performance and economic analysis of a single server feedback queueing model with vacation and impatient customers. *Opsearch* **56** (2019) 300–323.

[10] S.R Chakravarthy, An inventory system with Markovian demands, phase type distributions for perishability and replenishment. *Opsearch* **47** (2010) 266–283.

[11] S.R. Chakravarthy and B.M. Rao, Queuing-inventory models with random replenishment opportunities. *Mathematics* **9** (2021) 1092.

[12] S.R. Chakravarthy and A. Rumyantsev, Analytical and simulation studies of queueing-inventory models with MAP demands in batches and positive phase type services. *Simul. Model. Pract. Theory* **103** (2020) 102092.

[13] K.H. Choi, B.K. Yoon and S.A. Moon, Queueing inventory systems with phase-type service distributions: a literature review. *Ind. Eng. Manage. Syst.* **18** (2019) 330–339.

[14] H. Daduna, On queueing-inventory-location problems. *Ann. Oper. Res.* **331** (2023) 679–710.

[15] M.P. de Brito and E.A. van der Laan, Inventory control with product returns: the impact of imperfect information. *Eur. J. Oper. Res.* **194** (2009) 85–101.

[16] S. Dissa and P.V. Ushakumari, Two commodity queueing inventory system with random common lifetime, two demand classes and pool of customers. *Heliyon* **9** (2023) e21478.

[17] S. Dissa and P.V. Ushakumari, Two commodity perishable queueing inventory system with random common lifetime of commodities and positive lead time. *Opsearch* **61** (2024) 809–834.

[18] G. Hanukov, A queueing-inventory system with a repeated-orbit policy during the service. *Ann. Oper. Res.* (2023). DOI: 10.1007/s10479-023-05648-0.

[19] K. Jeganathan, S. Selvakumar, S. Saravanan, N. Anbazhagan, S. Amutha, W. Cho, G.P. Joshi and J. Ryoo, Performance of stochastic inventory system with a fresh item, returned item, refurbished item, and multi-class customers. *Mathematics* **10** (2022) 1137.

[20] K.P. Jose and S.S. Nair, Analysis of two production inventory systems with buffer, retrials and different production rates. *J. Ind. Eng. Int.* **13** (2017) 369–380.

[21] V.S. Koroliuk, L.A. Ponomarenko, A.Z. Melikov and A.M. Rustamov, Methods for analysis of multi-channel queueing models with instantaneous and delayed feedbacks. *Cyber. Syst. Anal.* **52** (2016) 58–70.

[22] A. Krishnamoorthy and K.P. Jose, Comparison of Inventory systems with service, positive lead-time, loss, and retrial of customers. *J. Appl. Math. Stoch. Anal.* **2007** (2007) 37848.

[23] A. Krishnamoorthy, B. Lakshmy and R. Manikandan, A survey on inventory models with positive service time. *Opsearch* **48** (2011) 153–169.

[24] A. Krishnamoorthy, D. Shajin and B. Lakshmy, On a queueing-inventory with reservation, cancellation, common life time and retrial. *Ann. Oper. Res.* **247** (2016) 365–389.

[25] A. Krishnamoorthy, D. Shajin and B. Lakshmy, $GI/M/1$ type queueing-inventory systems with postponed work, reservation, cancellation and common life time. *Indian J. Pure Appl. Math.* **47** (2016) 357–388.

[26] A. Krishnamoorthy, B. Benny and D. Shajin, A revisit to queueing-inventory system with reservation, cancellation and common life time. *Opsearch* **54** (2017) 336–350.

[27] A. Krishnamoorthy, D. Shajin and V.C. Narayanan, Inventory with positive service time: a survey, in Advanced Trends in Queueing Theory. *Series of Books Mathematics and Statistics? Sciences*, edited by V. Anisimov, N. Limnios. Vol. 2. ISTE & Wiley, London, UK (2021) 201–238.

[28] Z. Lian and M.F. Neuts, A discrete-time model for common lifetime inventory systems. *Math. Oper. Res.* **30** (2005) 718–732.

[29] R. Manikandan and S.S. Nair, $M/M/1/1$ queueing-inventory system with retrial of unsatisfied customers. *Commun. Appl. Anal.* **21** (2017) 217–236.

[30] A. Melikov and A. Molchanov, Stock optimization in transport/storage systems. *Cybernetics* **28** (1992) 484–487.

[31] A. Melikov, A. Krishnamoorthy and M. Shahmaliyev, Numerical analysis and long run total cost optimization of perishable queuing inventory systems with delayed feedback. *Queuing Models Serv. Manage.* **2** (2019) 83–111.

[32] M.F. Neuts, Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach. John Hopkins University Press, Baltimore, MD, USA (1981).

[33] S. Otten and H. Daduna, Stability of queueing-inventory systems with customers of different priorities. *Ann. Oper. Res.* **331** (2023) 963–983.

[34] K. Rasmi, M.J. Jacob, A. Dudin and A. Krishnamoorthy, Queueing inventory system with multiple service nodes and addressed retrials from a common orbit. *Methodol. Comput. Appl. Probab.* **26** (2024) 1–15.

[35] N. Saranya, S.A. Lawrence and B. Sivakumar, An inventory system with replacement and refurbishment of failed items. *Commun. Stat. Theory Methods* **52** (2023) 5966–5988.

[36] M. Schwarz and H. Daduna, Queuing systems with inventory management with random lead times and with backordering. *Math. Methods Oper. Res.* **64** (2006) 383–414.

[37] M. Schwarz, C. Sauer, H. Daduna, R. Kulik and R. Szekli, $M/M/1$ queuing systems with inventory. *Queuing Syst. Theory Appl.* **54** (2006) 55–78.

[38] D. Shajin and A. Krishnamoorthy, Stochastic decomposition in retrial queueing-inventory system. *RAIRO-Oper. Res.* **54** (2020) 81–99.

[39] D. Shajin and A. Krishnamoorthy, On a queueing-inventory system with impatient customers, advanced reservation, cancellation, overbooking and common life time. *Oper. Res. Int. J.* **21** (2021) 1229–1253.

[40] D. Shajin, B. Benny, T.G. Deepak and A. Krishnamoorthy, A relook at queueing-inventory system with reservation, cancellation and common life time. *Commun. Appl. Anal.* **20** (2016) 545–574.

[41] D. Shajin, A. Krishnamoorthy, A.N. Dudin, V.C. Joshua and J. Varghese, On a queueing-inventory system with advanced reservation and cancellation for the next K time frames ahead: the case of overbooking. *Queueing Syst.* **94** (2020) 3–37.

[42] D. Shajin, A. Krishnamoorthy and R. Manikandan, On a queueing-inventory system with common life time and Markovian lead time process. *Oper. Res. Int. J.* **22** (2022) 651–684.

[43] K. Sigman and D. Simchi-Levi, Light traffic heuristic for an $M/G/1$ queue with limited inventory. *Ann. Oper. Res.* **40** (1992) 371–380.

[44] C. Suganya, M. Amirthakodi and B. Sivakumar, An inventory system with two heterogeneous servers and orbital search for feedback customers. *Int. J. Math. Modell. Numer. Optim.* **8** (2018) 351–377.

[45] L. Takacs, A single-server queue with feedback. *Bell Syst. Tech. J.* **42** (1963) 505–519.

[46] L. Takacs, A queueing model with feedback. *Oper. Res.* **11** (1977) 345–354.

[47] V. Vinitha, N. Anbazhagan, S. Amutha, K. Jeganathan, G.P. Joshi, W. Cho and S. Seo, Steady state analysis of impulse customers and cancellation policy in queueing-inventory system. *Processes* **9** (2021) 2146.

Appendix A.

Sub-matrices are

$$L_0 = \begin{pmatrix} L_0^* & & & \\ & L_0^* & & \\ & & \ddots & \\ & & & L_0^* \end{pmatrix}, \quad L = \begin{pmatrix} I \otimes D_1 & & & \\ & I \otimes D_1 & & \\ & & \ddots & \\ & & & I \otimes D_1 \end{pmatrix},$$

$$L_0^* = \begin{pmatrix} \phi_1 D_1 & & & \\ & I \otimes D_1 & & \\ & & \ddots & \\ & & & I \otimes D_1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} M_0^* & & & \\ & M_0^* & & \\ & & \ddots & \\ & & & M_0^* \end{pmatrix},$$

$$M_1 = \begin{pmatrix} M_1^{(0)} & M_1^{(1)} & \dots & M_1^{(N)} \end{pmatrix}, \quad M = \begin{pmatrix} \mathbf{T}^0\boldsymbol{\alpha} \otimes I & & & \\ & \mathbf{T}^0\boldsymbol{\alpha} \otimes I & & \\ & & \ddots & \\ & & & \mathbf{T}^0\boldsymbol{\alpha} \otimes I \end{pmatrix},$$

$$M_0^* = \begin{pmatrix} \mathbf{0} & & & & \\ \mathbf{T}^0 \otimes I & & & & \\ & \mathbf{T}^0\boldsymbol{\alpha} \otimes I & & & \\ & & \ddots & & \\ & & & \mathbf{T}^0\boldsymbol{\alpha} \otimes I & \end{pmatrix}, \quad M_1^{(0)} = \begin{pmatrix} & \mathbf{T}^0\boldsymbol{\alpha} \otimes I \\ & \\ & \end{pmatrix}, \quad M_1^{(1)} = \begin{pmatrix} & \\ & \mathbf{T}^0\boldsymbol{\alpha} \otimes I \\ & \\ & \end{pmatrix},$$

$$M_1^{(N)} = \begin{pmatrix} & & \\ & & \\ & & \\ & \mathbf{T}^0\boldsymbol{\alpha} \otimes I & \end{pmatrix}, \quad G_0 = \begin{pmatrix} G_0^* & G_0^{**} & & & \\ & G_0^* & G_0^{**} & & \\ & & \ddots & \ddots & \\ & & & G_0^* & G_0^{**} \\ & & & & G_0'^{*} \end{pmatrix}, \quad G = \begin{pmatrix} G^{(0)} \\ G^{(1)} \\ \vdots \\ G^{(N)} \end{pmatrix},$$

$$G_0^* = \begin{pmatrix} u_0 & \sigma_r\zeta_r\boldsymbol{\alpha}\otimes I & & & & \theta\boldsymbol{\alpha}\otimes I \\ & u_1 & \sigma_r\zeta_r\otimes I & & & \theta\otimes I \\ & & \ddots & \ddots & & \vdots \\ & & & u_1 & \sigma_r\zeta_r\otimes I & \theta\otimes I \\ & & & & u_1 & (\sigma_r\zeta_r+\theta)\otimes I \\ & & & & & u_2 \end{pmatrix},$$

$$u_0 = \phi_2 D_1 + D_0 - (\sigma_r\zeta_r+\sigma_f\zeta_f+\theta)\otimes I, \quad u_1 = T\oplus D_0 - (\sigma_r\zeta_r+\sigma_f\zeta_f+\theta)\otimes I,$$
$$u_2 = T\oplus D_0 - (\sigma_r\zeta_r+\sigma_f\zeta_f)\otimes I,$$

$$G_0'^* = \begin{pmatrix} u_0' & \sigma_r\zeta_r\boldsymbol{\alpha}\otimes I & & & & \theta\alpha\otimes I \\ & u_1' & \sigma_r\zeta_r\otimes I & & & \theta\otimes I \\ & & \ddots & \ddots & & \vdots \\ & & & u_1' & \sigma_r\zeta_r\otimes I & \theta\otimes I \\ & & & & u_1' & (\sigma_r\zeta_r+\theta)\otimes I \\ & & & & & u_2' \end{pmatrix},$$

$$u_0' = \phi_2 D_1 + D_0 - (\sigma_r\zeta_r+\theta)\otimes I, u_1' = T\oplus D_0 - (\sigma_r\zeta_r+\theta)\otimes I, u_2' = T\oplus D_0 - \sigma_r\zeta_r\otimes I,$$

$$G_0^{**} = \begin{pmatrix} \sigma_f\zeta_f\otimes I & & & \\ & \sigma_f\zeta_f\otimes I & & \\ & & \ddots & \\ & & & \sigma_f\zeta_f\otimes I \end{pmatrix}, \quad G^{(0)} = \begin{pmatrix} & & \\ & & \\ & & \\ \sigma_r\zeta_r\otimes I & & \end{pmatrix}, \quad G^{(1)} = \begin{pmatrix} & & \\ & & \\ & & \\ & \sigma_r\zeta_r\otimes I & \end{pmatrix},$$

$$G^{(N)} = \begin{pmatrix} & & \\ & & \\ & & \\ & & \sigma_r\zeta_r\otimes I \end{pmatrix}, \quad G_2 = \begin{pmatrix} \sigma_r\zeta_r\otimes I & & & \\ & \sigma_r\zeta_r\otimes I & & \\ & & \ddots & \\ & & & \sigma_r\zeta_r\otimes I \end{pmatrix},$$

$$G_1 = \begin{pmatrix} v & \sigma_f\zeta_f\otimes I & & & \\ & v & \sigma_f\zeta_f\otimes I & & \\ & & \ddots & \ddots & \\ & & & v & \sigma_f\zeta_f\otimes I \\ & & & & v' \end{pmatrix}, \quad v = T\oplus D_0 - (\sigma_r\zeta_r+\sigma_f\zeta_f)\otimes I,$$

$$v' = T\oplus D_0 - \sigma_r\zeta_r\otimes I, \quad G_3 = \begin{pmatrix} v_0 & \sigma_f\zeta_f\otimes I & & & \\ & v_0 & \sigma_f\zeta_f\otimes I & & \\ & & \ddots & \ddots & \\ & & & v_0 & \sigma_f\zeta_f\otimes I \\ & & & & v_0' \end{pmatrix}, \quad v_0 = T\oplus D_0 - \sigma_f\zeta_f\otimes I,$$

$$v_0' = T\oplus D_0.$$

$$
\hat{L}_0 = \begin{pmatrix} \hat{L}_0^* & & & \\ \hat{L}_1 & \hat{L}_0^* & & \\ & & \ddots & \\ & & \hat{L}_N & \hat{L}_0^* \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} \boldsymbol{\alpha} \otimes D_1 & & & & \\ \eta\boldsymbol{\alpha} \otimes I & \boldsymbol{\alpha} \otimes D_1 & & & \\ & 2\eta\boldsymbol{\alpha} \otimes I & \boldsymbol{\alpha} \otimes D_1 & & \\ & & \ddots & \ddots & \\ & & & N\eta\boldsymbol{\alpha} \otimes I & \boldsymbol{\alpha} \otimes D_1 \end{pmatrix},
$$

$$
\hat{L}_0^* = \begin{pmatrix} \phi_1 D_1 & & & \\ & \boldsymbol{\alpha} \otimes D_1 & & \\ & & \ddots & \\ & & & \boldsymbol{\alpha} \otimes D_1 \end{pmatrix}, \quad \hat{L}_i = \begin{pmatrix} \mathbf{0} & & & \\ & i\eta\boldsymbol{\alpha} \otimes I & & \\ & & \ddots & \\ & & & i\eta\boldsymbol{\alpha} \otimes I \end{pmatrix}, \qquad 1 \le i \le N
$$

$$
\hat{M}_0 = \begin{pmatrix} \hat{M}_0^* & & & \\ & \hat{M}_0^* & & \\ & & \ddots & \\ & & & \hat{M}_0^* \end{pmatrix}, \quad \hat{M}_1 = \begin{pmatrix} \hat{M}_1^{(0)} & \hat{M}_1^{(1)} & \dots & \hat{M}_1^{(N)} \end{pmatrix},
$$

$$
\hat{M} = \begin{pmatrix} \mathbf{T}^0 \otimes I & & & \\ & \mathbf{T}^0 \otimes I & & \\ & & \ddots & \\ & & & \mathbf{T}^0 \otimes I \end{pmatrix}, \quad \hat{M}_0^* = \begin{pmatrix} \mathbf{0} & & & \\ \mathbf{T}^0 \otimes I & & & \\ & \mathbf{T}^0 \otimes I & & \\ & & \ddots & \\ & & & \mathbf{T}^0 \otimes I \end{pmatrix},
$$

$$
\hat{M}_1^{(0)} = \begin{pmatrix} & \mathbf{T}^0 \otimes I \\ & \\ & \end{pmatrix}, \quad \hat{M}_1^{(1)} = \begin{pmatrix} & \\ & \mathbf{T}^0 \otimes I \\ & \\ & \end{pmatrix}, \quad \hat{M}_1^{(N)} = \begin{pmatrix} & \\ & \\ & \mathbf{T}^0 \otimes I \end{pmatrix},
$$

$$
\hat{G}_0 = \begin{pmatrix} \hat{G}_{00}^* & \hat{G}_0^{**} & & & \\ & \hat{G}_{01}^* & \hat{G}_0^{**} & & \\ & & \ddots & \ddots & \\ & & & \hat{G}_{0N-1}^* & \hat{G}_0^{**} \\ & & & & \hat{G}_{0N}'^* \end{pmatrix}, \quad \hat{G} = \begin{pmatrix} \hat{G}^{(0)} \\ \hat{G}^{(1)} \\ \vdots \\ \hat{G}^{(N)} \end{pmatrix},
$$

$$
\hat{G}_{0i}^* = \begin{pmatrix} \hat{u}_0 & \sigma_r\zeta_r \otimes I & & & & \theta \otimes I \\ & \hat{u}_{1i} & \sigma_r\zeta_r \otimes I & & & \theta \otimes I \\ & & \ddots & \ddots & & \vdots \\ & & & \hat{u}_{1i} & \sigma_r\zeta_r \otimes I & \theta \otimes I \\ & & & & \hat{u}_{1i} & (\sigma_r\zeta_r + \theta) \otimes I \\ & & & & & \hat{u}_{2i} \end{pmatrix}, \qquad 0 \le i \le N-1
$$

$$
\hat{u}_0 = \phi_2 D_1 + D_0 - (\sigma_r\zeta_r + \sigma_f\zeta_f + \theta) \otimes I, \quad \hat{u}_{1i} = T \oplus D_0 - (i\eta + \sigma_r\zeta_r + \sigma_f\zeta_f + \theta) \otimes I,
$$
$$
\hat{u}_{2i} = T \oplus D_0 - (i\eta + \sigma_r\zeta_r + \sigma_f\zeta_f) \otimes I,
$$

$$\hat{G}'^{*}_{0N} = \begin{pmatrix} \hat{u}'_0 & \sigma_r\zeta_r \otimes I & & & & \theta \otimes I \\ & \hat{u}'_{1N} & \sigma_r\zeta_r \otimes I & & & \theta \otimes I \\ & & \ddots & \ddots & & \vdots \\ & & & \hat{u}'_{1N} & \sigma_r\zeta_r \otimes I & \theta \otimes I \\ & & & & \hat{u}'_{1N} & (\sigma_r\zeta_r + \theta) \otimes I \\ & & & & & \hat{u}'_{2N} \end{pmatrix},$$

$$\hat{u}'_0 = \phi_2 D_1 + D_0 - (\sigma_r\zeta_r + \theta) \otimes I, \hat{u}'_{1N} = T \oplus D_0 - (N\eta + \sigma_r\zeta_r + \theta) \otimes I, \hat{u}'_{2N} = T \oplus D_0 - (N\eta + \sigma_r\zeta_r) \otimes I,$$

$$\hat{G}^{**}_0 = \begin{pmatrix} \sigma_f\zeta_f \otimes I & & & \\ & \sigma_f\zeta_f \otimes I & & \\ & & \ddots & \\ & & & \sigma_f\zeta_f \otimes I \end{pmatrix}, \quad \hat{G}^{(0)} = \begin{pmatrix} & & & & \\ & & & & \\ \sigma_r\zeta_r \otimes I & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix},$$

$$\hat{G}^{(1)} = \begin{pmatrix} & & & & \\ & & & & \\ \mathbf{0} & \sigma_r\zeta_r \otimes I & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \quad \hat{G}^{(N)} = \begin{pmatrix} & & & & \\ & & & & \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \sigma_r\zeta_r \otimes I \end{pmatrix},$$

$$\hat{G}_2 = \begin{pmatrix} \sigma_r\zeta_r \otimes I & & & \\ & \sigma_r\zeta_r \otimes I & & \\ & & \ddots & \\ & & & \sigma_r\zeta_r \otimes I \end{pmatrix}, \quad \hat{G}_1 = \begin{pmatrix} \hat{v} & \sigma_f\zeta_f \otimes I & & & \\ & \hat{v}_1 & \sigma_f\zeta_f \otimes I & & \\ & & \ddots & \ddots & \\ & & & \hat{v}_{N-1} & \sigma_f\zeta_f \otimes I \\ & & & & \hat{v}'_N \end{pmatrix},$$

$$\hat{v} = T \oplus D_0 - (\sigma_r\zeta_r + \sigma_f\zeta_f) \otimes I, \hat{v}'_N = T \oplus D_0 - (N\eta + \sigma_r\zeta_r) \otimes I,$$

$$\hat{v}_i = T \oplus D_0 - (i\eta + \sigma_r\zeta_r) \otimes I, 1 \le i \le N-1, \hat{G}_3 = \begin{pmatrix} \hat{v}_{00} & \sigma_f\zeta_f \otimes I & & & \\ & \hat{v}_{01} & \sigma_f\zeta_f \otimes I & & \\ & & \ddots & \ddots & \\ & & & \hat{v}_{0N-1} & \sigma_f\zeta_f \otimes I \\ & & & & \hat{v}'_{0N} \end{pmatrix},$$

$$\hat{v}_{0i} = T \oplus D_0 - (i\eta + \sigma_f\zeta_f) \otimes I, 0 \le i \le N-1, \hat{v}'_{0N} = T \oplus D_0 - N\eta \otimes I.$$

Appendix B.

The sub-matrices are

$$\mathcal{L}_0 = \begin{pmatrix} L' & & \\ & \ddots & \\ & & L' \end{pmatrix}, \quad L' = \begin{pmatrix} \phi_1\lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix}, \quad \mathcal{M}_0 = \begin{pmatrix} M' & & \\ & \ddots & \\ & & M' \end{pmatrix}, \quad M' = \begin{pmatrix} 0 & & & \\ \mu & & & \\ & \ddots & & \\ & & \mu & \end{pmatrix},$$

$$\mathcal{M}_1 = \begin{pmatrix} M_1^{(0)} & M_1^{(1)} & \dots & M_1^{(N)} \end{pmatrix}, \quad M_1^{(0)} = \begin{pmatrix} & & \mu \\ & & \\ & & \\ & & \end{pmatrix}, \quad M_1^{(1)} = \begin{pmatrix} & & \\ & & \mu \\ & & \\ & & \end{pmatrix},$$

$$M_1^{(N)} = \begin{pmatrix} & & \\ & & \\ & & \mu \end{pmatrix}, \quad \mathcal{G}_{00} = \begin{pmatrix} \tilde{G}_0^* & \sigma_f\zeta_f I & & & \\ & \tilde{G}_0^* & \sigma_f\zeta_f I & & \\ & & \ddots & \ddots & \\ & & & \tilde{G}_0^* & \sigma_f\zeta_f I \\ & & & & \tilde{G}_0'^* \end{pmatrix},$$

$$\mathcal{G}_0 = \begin{pmatrix} \tilde{G}^{(0)} \\ \tilde{G}^{(1)} \\ \vdots \\ \tilde{G}^{(N)} \end{pmatrix}, \quad \tilde{G}_0^* = \begin{pmatrix} \tilde{u}_0 & \sigma_r\zeta_r & & & & \theta \\ & \tilde{u}_1 & \sigma_r\zeta_r & & & \theta \\ & & \ddots & \ddots & & \vdots \\ & & & \tilde{u}_1 & \sigma_r\zeta_r & \theta \\ & & & & \tilde{u}_1 & (\sigma_r\zeta_r + \theta) \\ & & & & & \tilde{u}_2 \end{pmatrix},$$

$$\tilde{u}_0 = -(\phi_1\lambda + \sigma_r\zeta_r + \sigma_f\zeta_f + \theta), \quad \tilde{u}_1 = -(\lambda + \mu + \sigma_r\zeta_r + \sigma_f\zeta_f + \theta), \quad \tilde{u}_2 = -(\lambda + \mu + \sigma_r\zeta_r + \sigma_f\zeta_f),$$

$$\tilde{G}_0'^* = \begin{pmatrix} \tilde{u}_0' & \sigma_r\zeta_r & & & & \theta \\ & \tilde{u}_1' & \sigma_r\zeta_r & & & \theta \\ & & \ddots & \ddots & & \vdots \\ & & & \tilde{u}_1' & \sigma_r\zeta_r & \theta \\ & & & & \tilde{u}_1' & (\sigma_r\zeta_r + \theta) \\ & & & & & \tilde{u}_2' \end{pmatrix},$$

$$\tilde{u}_0' = -(\phi_1\lambda + \sigma_r\zeta_r + \theta), \quad \tilde{u}_1' = -(\lambda + \mu + \sigma_r\zeta_r + \theta), \quad \tilde{u}_2' = -(\lambda + \mu + \sigma_r\zeta_r),$$

$$\tilde{G}^{(0)} = \begin{pmatrix} & & & & \\ & & & & \\ \sigma_r\zeta_r & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \tilde{G}^{(1)} = \begin{pmatrix} & & & & \\ & & & & \\ 0 & \sigma_r\zeta_r & 0 & \dots & 0 \end{pmatrix}, \quad \tilde{G}^{(N)} = \begin{pmatrix} & & & & \\ & & & & \\ 0 & 0 & \dots & 0 & \sigma_r\zeta_r \end{pmatrix},$$

$$\mathcal{G}_1 = \begin{pmatrix} \tilde{v} & \sigma_f\zeta_f & & & \\ & \tilde{v} & \sigma_f\zeta_f & & \\ & & \ddots & \ddots & \\ & & & \tilde{v} & \sigma_f\zeta_f \\ & & & & \tilde{v}' \end{pmatrix}, \quad \tilde{v} = -(\lambda + \mu + \sigma_r\zeta_r + \sigma_f\zeta_f), \quad \tilde{v}' = -(\lambda + \mu + \sigma_r\zeta_r),$$

$$\mathcal{G}_1' = \begin{pmatrix} \tilde{v}_0 & \sigma_f\zeta_f & & & \\ & \tilde{v}_0 & \sigma_f\zeta_f & & \\ & & \ddots & \ddots & \\ & & & \tilde{v}_0 & \sigma_f\zeta_f \\ & & & & \tilde{v}_0' \end{pmatrix}, \quad \tilde{v}_0 = -(\lambda + \mu + \sigma_f\zeta_f), \quad \tilde{v}_0' = -(\lambda + \mu),$$

$$\hat{\mathcal{L}}_0 = \begin{pmatrix} L' & & & & \\ \hat{N}_1 & L' & & & \\ & \hat{N}_2 & L' & & \\ & & \ddots & \ddots & \\ & & & \hat{N}_N & L' \end{pmatrix}, \quad \hat{N}_i = \begin{pmatrix} 0 & & & \\ & i\eta & & \\ & & \ddots & \\ & & & i\eta \end{pmatrix}, \quad 1 \le i \le N, \quad \hat{\mathcal{L}} = \begin{pmatrix} \lambda & & & & \\ \eta & \lambda & & & \\ & 2\eta & \lambda & & \\ & & \ddots & \ddots & \\ & & & N\eta & \lambda \end{pmatrix},$$

$$\hat{\mathcal{G}}_{00} = \begin{pmatrix} \tilde{G}^*_{00} & \sigma_f\zeta_f I & & & \\ & \tilde{G}^*_{01} & \sigma_f\zeta_f I & & \\ & & \ddots & \ddots & \\ & & & \tilde{G}^*_{0N-1} & \sigma_f\zeta_f I \\ & & & & \tilde{G}'^*_{0N} \end{pmatrix},$$

$$\tilde{G}^*_{0i} = \begin{pmatrix} \tilde{u}_0 & \sigma_r\zeta_r & & & & \theta \\ & \tilde{u}_{i1} & \sigma_r\zeta_r & & & \theta \\ & & \ddots & \ddots & & \vdots \\ & & & \tilde{u}_{i1} & \sigma_r\zeta_r & \theta \\ & & & & \tilde{u}_{i1} & (\sigma_r\zeta_r+\theta) \\ & & & & & \tilde{u}_{i2} \end{pmatrix}, \quad 0 \le i \le N-1$$

$$\tilde{u}_0 = -(\phi_1\lambda + \sigma_r\zeta_r + \sigma_f\zeta_f + \theta), \quad \tilde{u}_{i1} = -(i\eta + \lambda + \mu + \sigma_r\zeta_r + \sigma_f\zeta_f + \theta), \quad \tilde{u}_{i2} = -(i\eta + \lambda + \mu + \sigma_r\zeta_r + \sigma_f\zeta_f),$$

$$\tilde{G}'^*_{0N} = \begin{pmatrix} \tilde{u}'_{N0} & \sigma_r\zeta_r & & & & \theta \\ & \tilde{u}'_{N1} & \sigma_r\zeta_r & & & \theta \\ & & \ddots & \ddots & & \vdots \\ & & & \tilde{u}'_{N1} & \sigma_r\zeta_r & \theta \\ & & & & \tilde{u}'_{N1} & (\sigma_r\zeta_r+\theta) \\ & & & & & \tilde{u}'_{N2} \end{pmatrix}, \quad \tilde{u}'_{N0} = -(\phi_1\lambda + \sigma_r\zeta_r + \theta),$$

$$\tilde{u}'_{N1} = -(N\eta + \lambda + \mu + \sigma_r\zeta_r + \theta), \quad \tilde{u}'_{N2} = -(N\eta + \lambda + \mu + \sigma_r\zeta_r), \quad \hat{\mathcal{G}}_1 = \begin{pmatrix} \tilde{v}_0 & \sigma_f\zeta_f & & & \\ & \tilde{v}_1 & \sigma_f\zeta_f & & \\ & & \ddots & \ddots & \\ & & & \tilde{v}_{N-1} & \sigma_f\zeta_f \\ & & & & \tilde{v}'_N \end{pmatrix},$$

$$\tilde{v}_i = -(i\eta + \lambda + \mu + \sigma_r\zeta_r + \sigma_f\zeta_f), \quad 0 \le i \le N-1, \quad \tilde{v}'_N = -(N\eta + \lambda + \mu + \sigma_r\zeta_r),$$

$$\mathcal{G}'_1 = \begin{pmatrix} \tilde{v}_{00} & \sigma_f\zeta_f & & & \\ & \tilde{v}_{01} & \sigma_f\zeta_f & & \\ & & \ddots & \ddots & \\ & & & \tilde{v}_{0N-1} & \sigma_f\zeta_f \\ & & & & \tilde{v}'_{0N} \end{pmatrix}, \quad \tilde{v}_{0i} = -(i\eta + \lambda + \mu + \sigma_f\zeta_f), \quad 0 \le i \le N-1,$$

$$\tilde{v}'_{0N} = -(N\eta + \lambda + \mu).$$

Appendix C.

$$\sigma_r\zeta_r\tilde{\boldsymbol{\pi}}_{K-1}I + \tilde{\boldsymbol{\pi}}_K\mathcal{H}_2 = 0 \Longrightarrow \tilde{\boldsymbol{\pi}}_{K-1} = \tilde{\boldsymbol{\pi}}_K\left(-\frac{\mathcal{H}_2}{\sigma_r\zeta_r}\right) = \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{U}}_{K-1}$$

$$\sigma_r\zeta_r\tilde{\boldsymbol{\pi}}_{K-2}I + \tilde{\boldsymbol{\pi}}_{K-1}\mathcal{H}_1 + \mu\tilde{\boldsymbol{\pi}}_K I = 0 \Longrightarrow \tilde{\boldsymbol{\pi}}_{K-2} = \tilde{\boldsymbol{\pi}}_K\left(-\frac{\tilde{\mathcal{U}}_{K-1}H_1 + \mu I}{\sigma_r\zeta_r}\right) = \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{U}}_{K-2}$$

$$\sigma_r\zeta_r\tilde{\boldsymbol{\pi}}_{K-3}I + \tilde{\boldsymbol{\pi}}_{K-2}\mathcal{H}_1 + \mu\tilde{\boldsymbol{\pi}}_{K-1}I = 0 \Longrightarrow \tilde{\boldsymbol{\pi}}_{K-3} = \tilde{\boldsymbol{\pi}}_K\left(-\frac{\tilde{\mathcal{U}}_{K-2}H_1 + \mu\tilde{\mathcal{U}}_{K-1}}{\sigma_r\zeta_r}\right) = \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{U}}_{K-3}.$$

Proceeding like this we get

$$\sigma_r\zeta_r\tilde{\boldsymbol{\pi}}_1 I + \tilde{\boldsymbol{\pi}}_2\mathcal{H}_1 + \mu\tilde{\boldsymbol{\pi}}_3 I = 0 \Longrightarrow \tilde{\boldsymbol{\pi}}_1 = \tilde{\boldsymbol{\pi}}_K\left(-\frac{\tilde{\mathcal{U}}_2 H_1 + \mu\tilde{\mathcal{U}}_3}{\sigma_r\zeta_r}\right) = \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{U}}_1$$

$$\tilde{\boldsymbol{\pi}}_0\mathcal{H}_0 + \tilde{\boldsymbol{\pi}}_1\mathcal{M}_1 = 0 \Longrightarrow \tilde{\boldsymbol{\pi}}_0 = \tilde{\boldsymbol{\pi}}_K\left(-\tilde{\mathcal{U}}_1\mathcal{M}_1\right)(\mathcal{H}_0)^{-1} = \tilde{\boldsymbol{\pi}}_K\tilde{\mathcal{U}}_0$$

$$\tilde{\boldsymbol{\pi}}\mathbf{e} = 1 \Longrightarrow \tilde{\boldsymbol{\pi}}_K\left[I + \tilde{\mathcal{U}}_{K-1} + \ldots + \tilde{\mathcal{U}}_0\right]\mathbf{e} = 1 \Longrightarrow \tilde{\boldsymbol{\pi}}_K\left[I + \sum_{j=0}^{K-1}\tilde{\mathcal{U}}_j\right]\mathbf{e} = 1.$$

See the matrices $\tilde{A}_0, \tilde{A}_2$ given in Section 5 and $\mathcal{L}_0, \mathcal{M}_0, \mathcal{M}_1$ in Appendix B.

$$\tilde{\boldsymbol{\pi}}\tilde{A}_0\mathbf{e} = \tilde{\boldsymbol{\pi}}_0\mathcal{L}_0\mathbf{e} + \lambda\sum_{j=1}^{K}\tilde{\boldsymbol{\pi}}_j\mathbf{e} = \tilde{\boldsymbol{\pi}}_0\mathcal{L}_0\mathbf{e} + \lambda\tilde{\boldsymbol{\pi}}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}$$

and

$$\begin{aligned}\tilde{\boldsymbol{\pi}}\tilde{A}_2\mathbf{e} &= \tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} + \tilde{\boldsymbol{\pi}}_1\mathcal{M}_1\mathbf{e} + \mu\sum_{j=2}^{K}\boldsymbol{\pi}_j\mathbf{e}\\ &= \tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} + \mu\tilde{\boldsymbol{\pi}}_1\mathbf{e} + \mu\boldsymbol{\pi}_K\left[\sum_{j=2}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}\\ &= \tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} + \mu\boldsymbol{\pi}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}.\end{aligned}$$

From the matrix $\mathcal{L}_0$ we have $\tilde{\boldsymbol{\pi}}_0\mathcal{L}_0\mathbf{e} = \phi_1\lambda\sum_{n_0=0}^{N}\tilde{\pi}(0,n_0,0) + \lambda\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i)$

$$\Longrightarrow \tilde{\boldsymbol{\pi}}_0\mathcal{L}_0\mathbf{e} = \lambda\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i) \text{ if } \phi_1 = 0.$$

Hence $\tilde{\boldsymbol{\pi}}\tilde{A}_0\mathbf{e} = \lambda\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i) + \lambda\tilde{\boldsymbol{\pi}}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}$ if $\phi_1 = 0$

$$\Longrightarrow \tilde{\boldsymbol{\pi}}\tilde{A}_0\mathbf{e} = \lambda\left(\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i) + \tilde{\boldsymbol{\pi}}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}\right).$$

From the matrix $\mathcal{M}_0$, $\tilde{\boldsymbol{\pi}}_0\mathcal{M}_0\mathbf{e} = \mu\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i)$.
So $\tilde{\boldsymbol{\pi}}\tilde{A}_2\mathbf{e} = \mu\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i) + \mu\boldsymbol{\pi}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}$

$$\Longrightarrow \tilde{\boldsymbol{\pi}}\tilde{A}_2\mathbf{e} = \mu\left(\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i) + \boldsymbol{\pi}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}\right) \quad \text{if } \phi_1 = 0.$$

Let $\mathcal{C} = \left(\sum_{n_0=0}^{N}\sum_{i=1}^{S}\tilde{\pi}(0,n_0,i) + \boldsymbol{\pi}_K\left[\sum_{j=1}^{K-1}\tilde{\mathcal{U}}_j + I\right]\mathbf{e}\right)$ then

$$\tilde{\boldsymbol{\pi}}\tilde{A}_0\mathbf{e} < \tilde{\boldsymbol{\pi}}\tilde{A}_2\mathbf{e} \Longrightarrow \lambda\mathcal{C} < \mu\mathcal{C} \Longrightarrow \lambda < \mu \quad \text{if } \phi_1 = 0.$$