Annaghili et al. *Boundary Value Problems* (2025) 2025:38
https://doi.org/10.1186/s13661-025-02027-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Spectral analysis for the almost periodic quadratic pencil with impulse

Shams Annaghili[1], Rakib Efendiev[1], Davron Aslonqulovich Juraev[2,3*] and Mohamed Abdalla[4]

*Correspondence:
juraevdavron12@gmail.com
[2]Scientific Research Center, Department of Mathematics, Baku Engineering University, Baku, AZ0102, Azerbaijan
[3]Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, Uzbekistan
Full list of author information is available at the end of the article

Abstract

In this study, we delve into the spectral properties of a pencil of nonself-adjoint second-order differential operators characterized by almost periodic potentials and impulse conditions. Such operators arise in various physical models, particularly in quantum mechanics, where they describe systems with discontinuities in their potentials or boundary conditions. Understanding the spectrum of these operators is crucial for comprehending the stability and dynamics of the associated physical systems. By investigating the spectral gaps and accumulation points we aim to contribute to the broader understanding of non-self-adjoint operator theory and its applications in mathematical physics.

Mathematics Subject Classification: 34A12; 34A37

Keywords: Schrödinger equation; Impulse condition; Besikovich almost-periodic functions; Jost solution; Spectral singularities; Inverse problem

1 Introduction

The study of spectral properties of the impulse Schrödinger equation is pivotal for understanding quantum systems influenced by external forces. By examining the eigenvalues and eigenfunctions we can discern how perturbations affect the stability and dynamics of quantum states. In particular, the presence of impulse actions introduces unique challenges that necessitate a careful analysis of the underlying mathematical framework. This analysis not only enriches our theoretical understanding but also has practical implications in fields such as quantum mechanics and condensed matter physics. Moreover, the inverse problem associated with the impulse Schrödinger equation seeks to reconstruct the potential or perturbative influence from the observable spectral data. This aspect of the study is crucial, as it connects theoretical predictions with empirical observations. Solving the inverse problem requires sophisticated techniques combining spectral analysis and operator theory, often leveraging tools like the Riesz basis and transformations of the spectral data. In recent years, advancements in numerical methods and algorithms have facilitated the exploration of these spectral properties, allowing for a more nuanced understanding of complex quantum systems. Through simulation and computational modeling, researchers aim to identify the relationships between impulse phenomena and spectral

© The Author(s) 2025. **Open Access** This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by-nc-nd/4.0/.

characteristics, thereby broadening our comprehension of quantum behaviors under external perturbations.

In many practical scenarios, such as fluid dynamics, elasticity, and heat conduction, jump conditions arise at interfaces where two distinct materials meet. These interfaces can lead to abrupt changes in parameters such as density, thermal conductivity, or elasticity, affecting the behavior and propagation of waves, stress, or temperature fields [1]. The mathematical formulation of such problems often necessitates the use of specialized techniques, including the method of matched asymptotic expansions or the use of Green's functions tailored to accommodate the discontinuities present in the medium (see [2, 3]). Researchers have developed a variety of numerical methods to tackle boundary value problems characterized by jump conditions. Finite element methods (FEMs), for instance, have been adapted to enhance the treatment of discontinuities by employing enriched interpolation functions or interface tracking algorithms. Such advancements improve calculation accuracy and expand the applicability of numerical simulations to complex geometries and material compositions prevalent in engineering and physics. Furthermore, analytical solutions, although less common, provide valuable insights into the underlying mechanics of jump conditions. Perturbation techniques and integral transforms often reveal simplified models that establish benchmarks for validating numerical results. The synergy between analytical and numerical approaches facilitates a deeper understanding of the phenomena associated with discontinuous media, paving the way for innovative applications in materials science and engineering (see, for example, [1–19]).

This study focuses on the spectral characteristics and the inverse issues associated with the impulse Schrödinger equation. It seeks to explore the underlying properties of the spectrum while also addressing the challenges presented by the inverse problem. By examining these aspects the paper intends to provide a deeper understanding of the dynamics and implications of the impulse Schrödinger equation in various contexts. The research will contribute to the existing literature by shedding light on both the spectral analysis and the effective methodologies for tackling inverse problems, which have significant relevance in the field of quantum mechanics and mathematical physics. Overall, the objective is to enhance knowledge regarding this equation's behavior and its potential applications in theoretical and practical scenarios. Through rigorous investigation and analysis, this paper aspires to add valuable insights and potentially pave the way for further developments in the study of impulse Schrödinger equations.

We consider the equation

$$\begin{aligned} &-\eta'' + 2\mu p(\xi)\eta + q\left(\xi\right)\eta = \mu^2 \rho\left(\xi\right)\eta, \\ &\xi \in [0,\xi_0) \cup (\,\xi_0,\infty), \end{aligned} \tag{1}$$

in the space $L_2\,[0,\infty)$ with the potentials $p\,(\xi) \in P\,(\Upsilon)$and $q\,(\xi) \in Q\,(\Upsilon)$ where $P\,(\Upsilon)$ and $Q\,(\Upsilon)$ are Besikovich almost-periodic functions classes, which means that in the $L_2\,[0,\infty)$ domain, we consider potentials $p\,(\xi) \in P\,(\Upsilon)$ and $q\,(\xi) \in Q\,(\Upsilon)$, where the sets P($\Upsilon$) and Q($\Upsilon$) consist of Besikovich almost-periodic functions. This indicates that

$$P\,(\Upsilon) = \{\varrho : \varrho\,(\xi) = \sum_{p=1}^{\infty} \varrho_p e^{i\alpha_\upsilon \xi};\ \sum_{p=1}^{\infty} \alpha_p \left|\varrho_p\right| < \infty\}, \tag{2}$$

$$Q(\Upsilon) = \{\sigma : \sigma(\xi) = \sum_{p=1}^{\infty} \sigma_p e^{i\alpha_p \xi};\ \sum_{p=1}^{\infty} |\sigma_p| < \infty\}, \tag{3}$$

and

$$\rho(x) = \begin{cases} 1, & \xi \geq 0, \\ \beta^2, \beta \neq \pm 1, & \xi < 0, \end{cases} \tag{4}$$

assuming that $\Upsilon = \{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$, $\alpha_n > 0$, $p \in N$, is a finite set of positive real numbers that is closed under addition.

For this purpose, we consider the operator

$$L = \frac{1}{\rho(\xi)} \left[-\frac{d^2}{d\xi^2} + 2\mu\varrho(\xi) + \sigma(\xi) \right]$$

generated by equation (16), the boundary condition

$$\eta(0) = 0, \tag{5}$$

and defined for complex numbers α_i, $i = \overline{1,4}$, the momentum condition

$$\begin{bmatrix} \eta(\xi_0^+) \\ \eta'(\xi_0^+) \end{bmatrix} = B \begin{bmatrix} \eta(\xi_0^-) \\ \eta'(\xi_0^-) \end{bmatrix}, B = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \det B \neq 0. \tag{6}$$

The point $\xi = \xi_0$ is referred to as the impulse point of problem (1) with matrix B facilitating the extension of the solution for equation (1) from the interval $[0, x_0)$ to the interval (x_0, ∞).

In the frequency domain, equation (1) captures the dynamics of wave propagation in a one-dimensional nonconservative medium. Here μ signifies momentum, μ^2 represents energy, $\varrho(\xi)$ Illustrates the combined effects of energy absorption and generation, and $\sigma(\xi)$ pertains to the regeneration of force density.

Typically, the issue at hand is connected to discontinuities in the physical properties of the medium [10].

Boundary value problems featuring discontinuities are commonly encountered in numerous physical contexts, especially when dealing with materials that exhibit nonuniform characteristics (refer to [4–7, 10, 12–16]). A wide array of scholars has conducted comprehensive research and further developed these types of issues (see, for instance, [7–9, 17–19]).

The study of inverse spectral problems is fundamentally tied to the reconstruction of differential operators from the spectral data associated with their eigenvalues and eigenfunctions. The central question driving this inquiry is whether we can uniquely determine the potential of a Sturm–Liouville operator given its spectrum. This problem has historical roots tracing back to work by mathematicians such as S. Krein and M. Gelfand, who laid the groundwork for a rich field of research, leading to a deeper understanding of the relationships between analysis and geometry. One significant result in this domain is the complete characterization of certain classes of Sturm–Liouville problems, where the spectrum can indeed be shown to uniquely identify the associated potential. The methods employed often involve complex analysis, differential equations, and, increasingly, numerical techniques that yield practical algorithms for extraction and reconstruction of these operators. Furthermore, advances in inverse problems have implications reaching beyond pure mathematics, impacting physics, engineering, and even quantum mechanics, inviting interdisciplinary collaboration. Moreover, the complexity of inverse spectral problems extends to higher-order operators and more intricate boundary conditions, where the uniqueness of recovery is not always guaranteed. Recent research has been delving into these more complicated scenarios, revealing fascinating connections to topology and the overarching geometry of spectral manifolds. As the field progresses, it continues to pose significant challenges and stimulate innovative approaches, paving the way for future discoveries in mathematical physics.

It is important to mention that Gasymov [11] examined the spectral properties associated with the case $L\eta = \mu^2\eta$ at $\varrho(\xi) = 0$, $\rho(\xi) \equiv 1$, $\alpha_p = p$, provided that condition (3) holds. Meanwhile, Orudzhev [20] addressed the scenario where $\varrho(\xi) = 0$ and $\rho(\xi) \equiv 1$, also under condition (3). Additionally, several boundary value problems have been analyzed in [20–28]. The case $\alpha_p = p$, $p \in N$, was considered in [12]. Finally, we note that the operator generated by a finite sum in (3) for $p(x) = 0$ and $\rho(x) \equiv 1$ was studied by Sarnak [22].

Further, we will write $p \gg \nu$ or $p \ll \nu$ if $\alpha_p > \alpha_\nu$ or $\alpha_p < \alpha_\nu$, respectively. The symbol $\sum_{p:p>\nu}$ will be used for summing over all p such that $\alpha_p > \alpha_\nu$. We also will write $p \oplus \nu = \gamma$ if $\alpha_p + \alpha_\nu = \alpha_\gamma$.

For any μ_0, the limit

$$\angle \lim_{\mu\to\mu_0} f(\xi,\mu)(\mu-\mu_0) = \begin{cases} 0, \mu \notin \Upsilon, \\ f_p(\xi), \mu_0 = \alpha_p, p \in \Upsilon, \end{cases}$$

exists and is uniform in ξ.

In the subsequent discussion, the notation $\angle\lim$ signifies that the limit is taken in a nontangential direction as μ tends to μ_0 in such a way that for any specified $\delta > 0$, we have the inequality

$$\delta < \arg(\mu-\mu_0) < \pi - \delta.$$

The functions $\varrho(\xi)$ and $\sigma(\xi)$ will be referred to as the potentials associated with the equation

$$-\eta'' + 2\mu\varrho(\xi)\eta + \sigma(\xi)\eta = \mu^2\rho(\xi)\eta$$

or with the operator L.

2 Particular solution to the equation $Ly = \lambda^2 y$

Our objective within this section is to investigate the solutions to problem (1)—(4).

Let us denote by $\eta_-(\xi)$ and $\eta_+(\xi)$ the solutions of (1), respectively, in the intervals $[0,\xi_0)$ and (ξ_0,∞):

$$\begin{cases} \eta_-(\xi) := \eta(\xi), 0 \le \xi < \xi_0, \\ \eta_+(\xi) := \eta(\xi), \xi > \xi_0. \end{cases}$$

It is widely recognized [19] that within the range $[0,\xi_0)$, equation (1) possesses solutions $\chi(\xi,\mu)$ that can be regarded as equivalent to the solution of the corresponding integral equation

$$\chi(\xi,\mu) = \frac{\sin\mu\xi}{\mu} + \int_0^{\xi} \frac{\sin\mu(\xi-\eta)}{\mu}\left[\sigma(\xi) + 2\mu\varrho(\xi)\right]\chi(\eta,\mu)\,d\eta,$$

and satisfy following conditions:

$$\chi(0,\mu) = 0, \chi'(0,\mu) = 1.$$

In the interval $[0,\xi_0)$, there exists a solution $\omega(\xi,\mu)$ that satisfies the conditions

$$\omega(0,\mu) = 1, \omega'(0,\mu) = 0.$$

It is important to observe that the functions $\omega(\xi,\mu)$ and $\chi(\xi,\mu)$ are entire with respect to the parameter μ. These functions fulfill the criteria set forth

$$W[\omega(\xi,\mu), \psi(\xi,\mu)] = -1, \mu \in C,$$

where by $\boldsymbol{W}[\boldsymbol{\eta_1},\boldsymbol{\eta_2}]$ is the Wronskian of the functions $\boldsymbol{\eta_1}$ and $\boldsymbol{\eta_2}$.

Since all numbers $\{\alpha_p\}_{p\in N}$ are positive, it is possible to directly construct particular solutions for (1).

Theorem 1 *Equation (1) with potentials $\varrho(\xi) \in P(\Upsilon)$ and $\sigma(\xi) \in Q(\Upsilon)$ in the interval (ξ_0,∞) has a particular solution of the form*

$$f^+(\xi,\mu) = e^{i\beta\mu\xi}\left(1 + \sum_{p=1}^{\infty} V_p^+ e^{i\alpha_p\xi} + \sum_{p=1}^{\infty}\frac{1}{\alpha_p + 2\beta\mu}\sum_{s=p}^{\infty} V_{ps}^+ e^{i\alpha_s\xi}\right)$$

for all $\mu \in \overline{C_+} = \{\mu \in C : \operatorname{Im}\mu > 0\}$ that satisfies the asymptotics

$$\angle\lim_{\xi\to\infty} f^+(\xi,\mu)\,e^{-i\mu\beta\xi} = 1.$$

It is obvious that for all $\lambda \in \overline{C_-} = \{\lambda \in C : \operatorname{Im}\mu < 0\}$, the function

$$f^-(\xi,\mu) = e^{-i\beta\mu\xi}\left(1 + \sum_{p=1}^{\infty} V_p^- e^{i\alpha_p\xi} + \sum_{p=1}^{\infty}\frac{1}{\alpha_p - 2\beta\mu}\sum_{s=p}^{\infty} V_{ps}^- e^{i\alpha_s\xi}\right)$$

also is a solution that satisfies the asymptotics

$$\angle\lim_{\xi\to\infty} f^-(\xi,\mu)\,e^{+i\mu\beta\xi} = 1.$$

Here the numbers $V_p^{\pm}$ and $V_{p\alpha}^{\pm}$ are defined by the following recurrent formulae:

$$\alpha^2 V_p^{\pm} + \alpha\sum_{p=1}^{\alpha} V_{p\alpha}^{\pm} + \sum_{s=1}^{\alpha-1}\left(\sigma_{\alpha-s}V_s^{\pm} \pm p_{\alpha-s}\sum_{p=1}^{s} V_{ps}^{\pm}\right) + \sigma_\alpha = 0, \tag{7}$$

$$\alpha(\alpha - p)V_{p\alpha}^{\pm} + \sum_{s=p}^{\alpha-1} (\sigma_{\alpha-s} \mp p \cdot \varrho_{\alpha-s})\, V_{ps}^{\pm} = 0, \tag{8}$$

$$\alpha V_{\alpha}^{\pm} \pm \sum_{s=1}^{\alpha-1} p_{\alpha-s} V_{s}^{\pm} \pm \varrho_{\alpha} = 0, \tag{9}$$

in which the series $\sum_{p=1}^{\infty} \frac{1}{p} \sum_{\alpha=p}^{\infty} \alpha \left| V_{p\alpha}^{\pm} \right|$ and $\sum_{p=1}^{\infty} p^2 \left| V_{p}^{\pm} \right|$ are convergent.

It is straightforward to confirm that the Wronskian associated with these solutions upholds the subsequent relationship

$$W[f^{+}(\xi,\mu), f^{-}(\xi,\mu)] = -2i\mu\beta, \mu \in R.$$

Note that the functions $f_p^{\pm}(\xi)$ defined as

$$f_p^{\pm}(\xi) = \angle \lim_{\mu \to \mp \frac{\alpha_p}{2\beta}} \left(\alpha_p \pm 2\beta\mu\right) f^{\pm}(\xi,\mu) = \sum_{s=p}^{\infty} V_{ps}^{\pm} e^{i\alpha_s \xi} e^{-i\frac{\alpha_p}{2}\xi}$$

also are solutions of equation (16) for $\mu \neq \pm \frac{\alpha_p}{2\beta}$.

Since $W[f_p^{\pm}(\xi), f^{\mp}\left(\xi, \mp\frac{\alpha_p}{2\beta}\right)] = 0$, we obtain that

$$f_n^{\pm}(\xi) = S_p^{\pm} f^{\mp}\left(\xi, \mp\frac{\alpha_p}{2\beta}\right). \tag{10}$$

Comparing these relations, we will see that

$$\boldsymbol{S_p^{\pm}} = \boldsymbol{V_{pp}^{\pm}}.$$

Simple calculations show the following relation for the derivatives of the functions (10):

$$f_p^{\pm'}(\xi) = S_p^{\pm} f^{\mp'}\left(\xi, \mp\frac{\alpha_p}{2\beta}\right). \tag{11}$$

3 Resolvent construction

To study the spectral characteristics of equations (1)–(4), let us construct its resolvent.

It is clear that the complete solutions of equation (1) for $\lambda \in R$ can be derived by using linearly independent solutions within the intervals $[0, x_0)$ and (x_0, ∞) in the following manner:

$$\eta(\xi,\mu) = \begin{cases} c_1(\xi)\,\omega(\xi,\mu) + c_2(\xi)\,\chi(\xi,\mu), 0 \le \xi < \xi_0, \\ c_3(\xi) f^{+}(\xi,\mu) + c_4(\xi) f^{-}(\xi,\mu), \xi_0 < \xi < \infty. \end{cases}$$

In this context, the coefficients are arranged so that the requirements specified in (5) and (6) are satisfied for the solution $\eta(\xi,\mu)$.

Now let us develop the resolvent for the operator pencil L. To achieve this, we will tackle the following problem in $L_2[0,\infty)$:

$$\begin{aligned} &-\eta'' + 2\mu\varrho(\xi)\eta + \varrho(\xi)\eta = \mu^2 \rho(\xi)\eta + f(\xi), \xi \neq \xi_0, \\ &\eta(0) = 0, \end{aligned}$$

where $f(\xi)$ is any function in $L_2[0,\infty)$.

To determine the coefficients $c_j(\xi)$ for $j=\overline{1,4}$, we will analyze the system

$$\begin{aligned}
&c_1'(\xi)\,\omega(\xi,\mu)+c_2'(\xi)\,\chi(\xi,\mu)=0,\\
&c_1'(\xi)\,\omega(\xi,\mu)+c_2'(\xi)\,\chi(\xi,\mu)=f(\xi).
\end{aligned}$$

Considering that $W[\omega(\xi,\mu),\chi(\xi,\mu)]=-1$, where $\mu\in C$, to determine $c_j(\xi)$ for $j=\overline{1,2}$, we can use the following equations:

$$\begin{aligned}
&c_1(\xi)=\int_0^{\xi^*}\chi(t,\mu)f(t)\,dt+c_{1,}\\
&c_2(\xi)=-\int_{\xi^*}^{\xi_0}\omega(t,\mu)f(t)\,dt+c_2.
\end{aligned}$$

Comparably, we can determine the coefficients $c_j(\xi)$ for $j=\overline{3,4}$ while considering that

$$W[f^+(\xi,\mu),f^-(\xi,\mu)]=-2i\mu\beta,\mu\in R,$$

from which we have

$$\begin{aligned}
&c_3(\xi)=-\tfrac{1}{2i\beta\mu}\int_{\xi_0}^{\xi^{**}}f^-(t,\mu)f(t)\,dt+c_{3,}\\
&c_4(\xi)=\tfrac{1}{2i\beta\lambda}\int_{\xi^{**}}^{\infty}f^+(t,\mu)f(t)\,dt+c_4.
\end{aligned}$$

Thus we derive the solution to equation (1)

$$\eta(\xi,\mu)=\begin{cases}\eta_-(\xi,\mu),0\le\xi<\xi_{0,}\\ \eta_+(\xi,\mu),\xi_0<\xi<\infty,\end{cases}$$

where

$$\begin{aligned}
&\eta_-(\xi,\mu)=\int_0^{\xi^*}\omega(\xi,\mu)\,\chi(t,\mu)f(t)\,dt-\\
&-\int_{x^*}^{x_0}\omega(t,\mu)\,\chi(\xi,\mu)f(t)\,dt+c_1\omega(\xi,\mu)+c_2\chi(\xi,\mu),\\
&\eta_+(\xi,\mu)=-\tfrac{1}{2i\beta\mu}\int_{\xi_0}^{\xi^{**}}f^-(t,\mu)f^+(\xi,\mu)f(t)\,dt+\\
&+\tfrac{1}{2i\beta\mu}\int_{\xi^{**}}^{\infty}f^+(t,\mu)f^-(\xi,\mu)f(t)\,dt+c_4f^-(\xi,\mu)+c_3f^+(\xi,\mu).
\end{aligned}$$

Applying the initial condition $\eta(0)=0$ leads us to conclude that $c_2=0$, while the requirement $\eta(\xi,\mu)\in L_2[0,\infty)$ results in $c_4=0$. Consequently, for the functions

$$G_-(\xi,t,\mu)=\begin{cases}\omega(t,\mu)\,\psi(\xi,\mu),\xi\le t,\\ \omega(\xi,\mu)\,\psi(t,\mu),\xi\ge t,\end{cases}$$

and

$$G_+(\xi,t,\mu)=-\frac{1}{2i\mu\beta}\begin{cases}f^+(\xi,\mu)f^-(t,\mu),\xi\le t,\\ f^+(t,\mu)f^-(\xi,\mu),\xi\ge t,\end{cases}$$

the overall resolution for the system of equations (1)–(4) can be formulated as

$$\eta(\xi,\mu)=\begin{cases}\int_0^{\xi_0}G_-(\xi,t,\mu)f(t)\,dt+c_1\omega(\xi,\mu),0\le\xi<\xi_{0,}\\ \int_{\xi_0}^{\infty}G_+(\xi,t,\mu)f(t)\,dt+c_3f^+(\xi,\mu),\xi_0<\xi<\infty,\end{cases}$$

or for the function

$$G(\xi,t,\mu)=\begin{cases}G_-(\xi,t,\mu),0\le\xi<\xi_0,\\ G_+(\xi,t,\mu),\xi>\xi_0,\end{cases}$$

as

$$\eta(\xi,\mu)=\int_0^\infty G(\xi,t,\mu)f(t)\,dt+\begin{cases}c_1\omega(\xi,\mu),0\le\xi<x_0,\\ c_3f^+(\xi,\mu),x_0<\xi<\infty.\end{cases}$$

To find the coefficients c_1 and c_3, we use the condition

$$\begin{bmatrix}\eta\left(\xi_0^+\right)\\ \eta'\left(\xi_0^+\right)\end{bmatrix}=B\begin{bmatrix}\eta\left(\xi_0^-\right)\\ \eta'\left(\xi_0^-\right)\end{bmatrix}.$$

Note that from the last relationship we have

$$\begin{bmatrix}\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt+c_3f^+(\xi_0,\mu)\\ c_3f^{+'}(\xi_0,\mu)\end{bmatrix}$$
$$=\begin{bmatrix}\alpha_1\ \alpha_2\\ \alpha_3\ \alpha_4\end{bmatrix}\begin{bmatrix}\int_0^\infty G(\xi_0,t,\lambda)f(t)\,dt+c_1\omega(\xi_0,\mu)\\ c_1\omega'(\xi_0,\mu)\end{bmatrix}.$$

Then we obtain

$$\begin{aligned}&\textstyle\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt+c_3f^+(\xi_0,\mu)=\\ &=\alpha_1\left[\textstyle\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt+c_1\omega(\xi_0,\mu)\right]+\alpha_2c_1\omega'(\xi_0,\mu),\\ &c_3f^{+'}(\xi_0,\mu)=\alpha_2[\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt+c_1\varphi(\xi_0,\mu)]+\alpha_3c_1\varphi'(\xi_0,\mu).\end{aligned}$$

The coefficients c_1 and c_3 are found from the following system of equations:

$$\begin{aligned}&c_3f^+(\xi_0,\lambda)-c_1\left[\alpha_1\omega(\xi_0,\lambda)+\alpha_1\omega'(\xi_0,\lambda)\right]=(\alpha_1-1)\textstyle\int_0^\infty G(\xi_0,t,\lambda)f(t)\,dt,\\ &c_3f^{+'}(\xi_0,\lambda)-c_1\left[\alpha_3\omega(\xi_0,\lambda)+\alpha_4\omega'(\xi_0,\lambda)\right]=(\alpha_3-1)\textstyle\int_0^\infty G(\xi_0,t,\lambda)f(t)\,dt.\end{aligned}$$

By using the Cramer rule for a system of equations we have

$$c_3=\frac{\begin{vmatrix}(\alpha_1-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt & [\alpha_1\omega(\xi_0,\mu)+\alpha_1\omega'(\xi_0,\mu)]\\ (\alpha_3-1)\int_0^\infty G(\xi_0,t,\lambda)f(t)\,dt & [\alpha_3\omega(\xi_0,\mu)+\alpha_4\omega'(\xi_0,\mu)]\end{vmatrix}}{-f^+(\xi_0,\mu)\left[\alpha_3\omega(\xi_0,\mu)+\alpha_4\omega'(\xi_0,\mu)\right]+f^{+'}(\xi_0,\mu)\left[\alpha_3\omega(\xi_0,\mu)+\alpha_4\omega'(\xi_0,\mu)\right]}=$$
$$=\frac{\begin{vmatrix}(\alpha_1-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt & [\alpha_1\varphi(\xi_0,\mu)+\alpha_1\varphi'(\xi_0,\mu)]\\ (\alpha_3-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,d & [\alpha_3\varphi(\xi_0,\mu)+\alpha_4\varphi'(\xi_0,\mu)]\end{vmatrix}}{\Upsilon(\mu)}.$$

Similarly, we can show that

$$c_1=\frac{\begin{vmatrix}f^+(\xi_0,\mu) & (\alpha_1-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt\\ f^{+'}(\xi_0,\mu) & (\alpha_3-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt\end{vmatrix}}{\Upsilon(\mu)},$$

where

$$\Upsilon(\mu) = -f^{+}(\xi_0,\mu)\left[\alpha_3\omega(\xi_0,\mu) + \alpha_4\omega'(\xi_0,\mu)\right] + f^{+'}(\xi_0,\mu)\left[\alpha_3\omega(\xi_0,\mu) + \alpha_4\omega'(\xi_0,\mu)\right].$$

Thus the overall resolution for issues (1)–(4) takes the form

$$\eta(\xi,\lambda) = \int_0^\infty G(\xi,t,\mu)f(t)\,dt + \frac{1}{\Upsilon(\mu)}\begin{cases} \begin{vmatrix} f^{+}(\xi_0,\lambda) & (\alpha_1-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt \\ f^{+'}(\xi_0,\lambda) & (\alpha_3-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt \end{vmatrix}\omega(\xi,\mu), \\ \quad 0\le\xi<\xi_0, \\ \begin{vmatrix} (\alpha_1-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,dt & [\alpha_1\varphi(\xi_0,\mu)+\alpha_1\varphi'(\xi_0,\mu)] \\ (\alpha_3-1)\int_0^\infty G(\xi_0,t,\mu)f(t)\,d & [\alpha_3\varphi(\xi_0,\mu)+\alpha_4\varphi'(\xi_0,\mu)] \end{vmatrix} f^{+}(\xi,\mu), \\ \quad \xi_0<\xi<\infty. \end{cases}$$

Given that when $\operatorname{Im}\mu = 0$ and $\mu = \pm\frac{\alpha_p}{2}$ with $p\in N$, the primary set of solutions for equation (1) consists of the functions $f^{+}(\xi,\mu)$ and $f^{-}(\xi,\mu)$, we deduce that

$$\eta(\xi,\mu) = C_1 e^{i\operatorname{Re}\mu\xi}\left(1+\sum_{p=1}^{\infty}V_n^{+}e^{i\alpha_p\xi}+\sum_{p=1}^{\infty}\frac{1}{\alpha_p+2\mu}\sum_{s=p}^{\infty}V_{ps}^{+}e^{i\alpha_s\xi}\right) + C_2 e^{-i\operatorname{Re}\mu\xi}\left(1+\sum_{p=1}^{\infty}V_n^{-}e^{i\alpha_p\xi}+\sum_{p=1}^{\infty}\frac{1}{\alpha_p-2\mu}\sum_{s=p}^{\infty}V_{ps}^{-}e^{i\alpha_s\xi}\right).$$

Consequently, the solution to equation (1) exists within the space $L_2[0,\infty)$ if and only if $C_1 = C_2 = 0$ due to the periodic nature of its primary components. This indicates that the operator pencil lacks purely real eigenvalues.

To establish that the residual spectrum of the operator pencil L is vacant, we investigate the function $g(\xi,\mu)\in L_2[0,\infty)$, which acts as a solution to the adjoint equation $L^{*}(\mu)=0$ for the parameter $\mu\in C$. Then

$$-g''(\xi,\mu) + [2\mu\varrho(\xi)+\sigma(\xi)]g(\xi,\mu) = \mu^2 g(\xi,\mu). \tag{12}$$

Since (12) is an equation of type (1), we get that the point spectrum $\sigma_\varrho(L^{*}) = 0$ or the residual spectrum $\sigma_r(L) = 0$. This indicates that the operator pencil's spectrum has a continuous part $\sigma(L) = \sigma_c(L)$ and the operator L^{-1} is defined on a dense subset of $L_2[0,\infty)$ for $\mu\in C$.

On the other hand, the points $\lambda = \pm\frac{\alpha_p}{2}$, $n\in N$, can exclusively be classified as simple points for the operator pencil L^{-1}. Given that the operator L lacks eigenvalues, these points do not exhibit any singularities. onsequently, the spectrum of the operator pencil is entirely comprised of a continuous spectrum. Therefore, this continuous spectrum spans the range $\{-\infty<\mu<\infty\}$ and $\Upsilon(\mu)\neq 0$ when $\operatorname{Im}\mu = 0$.

All these details serve as evidence for the following theorem.

Theorem 2 *The spectrum of the operator pencil L includes a finite set of eigenvalues characterized as solutions to the equation* $M(\mu) = 0$, *alongside a continuous spectrum that spans the entire real line* $\{-\infty<\mu<\infty\}$. *Within this continuous spectrum, there may ex-*

ist spectral singularities aligning with the values $\mu = \pm\frac{\alpha_p}{2}$, where p is a natural number ($p \in N$).

It is straightforward to confirm that the function

$$G(x,\lambda) = \begin{cases} \psi(\xi,\mu), \mu \to 0^+, \\ C(\mu) f^+(\xi,\mu) + D(\mu) f^-(\xi,\mu), \xi \to \infty, \end{cases}$$

where

$$C(\lambda) = \frac{i}{2\mu}\{f^{-'}(\mu_0,\mu)[\alpha_1\chi(\xi_0,\mu) + \alpha_2\chi'(\xi_0,\mu)]$$
$$-f^-(\xi_0,\mu)[\alpha_3\chi(\xi_0,\mu) + \alpha_4\chi'(\xi_0,\mu)]\},$$
$$D(\mu) = \frac{i}{2\mu}\{-f^{+'}(\xi_0,\mu)[\alpha_1\chi(\xi_0,\mu) + \alpha_2\chi'(\xi_0,\mu)]$$
$$+f^+(\xi_0,\mu)[\alpha_3\chi(\xi_0,\mu) + \alpha_4\chi'(\xi_0,\mu)]\},$$

represents a resolution for issues (1)—(6) along the real number line. In the scenario where ξ approaches infinity, we have

$$G(\xi,\mu) = C(\mu) f^+(\xi,\mu) + D(\mu) f^-(\xi,\mu), \xi \to \infty.$$

By dividing both sides of the final equation by the function $D(\mu)$ we derive the expression

$$U^+(\xi,\mu) = \frac{C(\mu)}{D(\mu)} f_1^+(\xi,\mu) + f_1^-(\xi,\mu), \operatorname{Im}\mu = 0.$$

We call the function $U^+(\xi,\mu)$ an eigenfunction of the considered problem (1)–(4).

4 Inverse problem

Definition 1 The function $S(\mu) = \frac{C(\mu)}{D(\mu)}$ is called as reflection coefficient of problem (1)–(4).

Inverse problem: Using the reflection coefficient $S(\mu)$, derive the potentials $\varrho(\xi) \in P(G)$ and $\sigma(\xi) \in Q(G)$.

It is important to understand that the process of formulating the potentials $\varrho(\xi)$ and $\sigma(\xi)$ is directly linked to determining the values of ϱ_p and σ_p. To accomplish this objective, we employ

$$\lim_{\mu \to \mp\frac{\alpha_p}{2\beta}} (\alpha_p \pm 2\mu\beta) f^{\pm(j)}(\xi,\mu) = S_p^{\pm} f^{\mp(j)}\left(\xi, \mp\frac{\alpha_p}{2\beta}\right),$$

obtained from relations (10)–(11). Then we have

$$
\begin{aligned}
&\lim_{\mu\to\frac{\alpha_p}{2\beta}}(\alpha_p-2\mu\beta)S(\mu)=\\
&=-\lim_{\mu\to\frac{\alpha_p}{2\beta}}(\alpha_p-2\mu\beta)\frac{f^{-'}(\xi_0,\mu)[\alpha_1\chi(\xi_0,\mu)+\alpha_2\chi'(\xi_0,\mu)]-f^{-}(\xi_0,\mu)[\alpha_3\chi(\xi_0,\mu)+\alpha_4\chi'(\xi_0,\mu)]}{f^{+'}(\xi_0,\lambda)[\alpha_1\chi(\xi_0,\mu)+\alpha_2\chi'(\xi_0,\mu)]-f^{+}(\xi_0,\mu)[\alpha_3\chi(\xi_0,\mu)+\alpha_4\chi'(\xi_0,\mu)]}=\\
&=-\frac{\lim_{\mu\to\frac{\alpha_p}{2\beta}}(\alpha_p-2\mu\beta)f^{-'}(\xi_0,\mu)[\alpha_1\chi(\xi_0,\mu)+\alpha_2\chi'(\xi_0,\mu)]-\lim_{\mu\to\frac{\alpha_p}{2\beta}}(\alpha_p-2\mu\beta)f^{-}(\xi_0,\mu)[\alpha_3\chi(\xi_0,\mu)+\alpha_4\chi'(\xi_0,\mu)]}{\lim_{\mu\to\frac{\alpha_p}{2\beta}}f^{+'}(\xi_0,\mu)[\alpha_1\chi(\xi_0,\mu)+\alpha_2\chi'(\xi_0,\mu)]-\lim_{\mu\to\frac{\alpha_p}{2\beta}}f^{+}(\xi_0,\mu)[\alpha_3\chi(\xi_0,\mu)+\alpha_4\chi'(\xi_0,\mu)]}=\\
&=-\frac{S_p^{-}f^{+'}\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\left[\alpha_1\chi\left(\xi_0,\frac{\alpha_p}{2\beta}\right)+\alpha_2\chi'\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\right]-S_p^{-}f^{+}\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\left[\alpha_3\chi\left(\xi_0,\frac{\alpha_p}{2\beta}\right)+\alpha_4\chi'\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\right]}{f^{+'}\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\left[\alpha_1\chi\left(\xi_0,\frac{\alpha_p}{2\beta}\right)+\alpha_2\chi'\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\right]-f^{+}\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\left[\alpha_3\chi\left(\xi_0,\frac{\alpha_p}{2\beta}\right)+\alpha_4\chi'\left(\xi_0,\frac{\alpha_p}{2\beta}\right)\right]}=-S_p^{-}.
\end{aligned}
$$

Analogously, we can show that

$$
\lim_{\mu\to-\frac{\alpha_p}{2\beta}}(\alpha_p+2\mu\beta)\frac{1}{S(\mu)}=-S_p^{+}.
$$

So by using the reflection coefficient we find all the numbers $S_p^{\pm}=V_{pp}^{\pm}$. Then again by using a relation, which can be easily obtained from (11),

$$
V_{\nu s+\mu}^{(\pm)}=V_{\nu\nu}^{(\pm)}\left(V_s^{(\mp)}+\sum_{p=1}^{s}\frac{V_{p\alpha}^{(\mp)}}{\alpha_p+\alpha_\nu}\right),\ \nu,s=1,2,3,\ldots,
$$

all numbers $V_{p\alpha}^{\pm}$ and $V_\alpha^{\pm}$ can be found effectively and uniquely.

Finally, we have the following theorem.

Theorem 3 *All numbers $V_{p\alpha}^{\pm}$, $p>\alpha$, and $V_\alpha^{(\mp)}$ can be found by using the numbers $V_{pp}^{\pm}$ effectively and uniquely.*

Proof Denote

$$
\xi=it,\mu=-i\tau,\eta(\xi)=Y(t).
$$

Then, using equation (1), we derive the following expression:

$$
-Y''(t)+2\tau\overline{\varrho}(it)Y(t)+\overline{\sigma}(it)Y(t)=\tau^2Y(t),\tag{13}
$$

where

$$
\overline{\varrho}(t)=i\varrho(it)=i\sum_{p=1}^{\infty}\varrho_{pk}e^{-\alpha_p t},\overline{\sigma}(t)=-\sigma(it)=-\sum_{p=1}^{\infty}\sigma_p e^{-\alpha_p t}.\tag{14}
$$

Consequently, we derive equation (13), whose potentials diminish exponentially as t_k approaches infinity for $k=1,2,3$. The method of analytical continuation enables us to extract the relevant outcomes for equation (1) from those obtained for (13).

The solution to equation (13) with the potentials defined in (14) is given by

$$
f_{\pm}(t,\tau)=e^{\pm i\tau t}\left(1+\sum_{p=1}^{\infty}V_p^{\pm}e^{-\alpha_p t}+\sum_{p=1}^{\infty}\sum_{\alpha=p}^{\infty}\frac{V_{p\alpha}^{\pm}}{i\alpha_p\pm 2\tau}e^{-\alpha_p t}\right),\tag{15}
$$

and the numbers $V_p^{\pm}, V_{p\alpha}^{\pm}$ are determined by the recurrent formulae (7)–(9). Then from (15) we obtain that

$$f_{\pm}(t,\tau) = \Omega^{\pm}(t)\, e^{\pm i\tau t} + \int_t^{\infty} K^{\pm}(t,u)\, e^{\pm i\tau u} du,$$

where $K^{\pm}(t,u)$, $\Omega^{\pm}(t)$ are determined as

$$K^{\pm}(t,u) = \frac{1}{2i}\sum_{p=1}^{\infty}\sum_{\alpha=p}^{\infty} V_{p\alpha}^{\pm} e^{-\alpha t}\cdot e^{-\frac{(u-t)\alpha_p}{2}}, \Omega^{\pm}(t) = 1 + \sum_{p=1}^{\infty} V_p^{\pm} e^{-\alpha_p t}.$$

After rewriting (11) as

$$\begin{gathered} \textstyle\sum_{\alpha=p}^{\infty} V_{p\alpha}^{\pm} e^{-\alpha t}\cdot e^{\frac{\alpha_p t}{2}} = V_{pp}^{\pm} = \\ = V_{pp}^{\pm} e^{-\alpha_p t/2}\left(1 + \textstyle\sum_{\nu=1}^{\infty} V_{\nu}^{\mp} e^{-\alpha_m t} + \sum_{\nu=1}^{\infty}\sum_{s=\nu}^{\infty} \frac{V_{\nu s}^{\mp}}{i(\alpha_\nu+\alpha_p)} e^{-\alpha_s t}\right) \end{gathered} \tag{16}$$

and denoting

$$z^{\pm}(t+s) = \sum_{\nu=1}^{\infty} V_m^{\pm} e^{-(t+s)\alpha_\nu/2},$$

we have the following Marchenko-type equation:

$$K^{\pm}(t,s) = \Omega^{\pm}(t_k)\, z^{\pm}(t+s) + \int_t^{\infty} K^{\mp}(t,u)\, z^{\pm}(u+s)\, du. \tag{17}$$

It is commonly understood from courses on ordinary differential equations that

$$\Omega^{\pm}(t) = e^{\mp i\int_{\xi}^{\infty}\varrho(t)dt}.$$

Using this fact, we have

$$\Omega^{+}(t)\cdot\Omega^{-}(t) = 1. \tag{18}$$

On the other hand, from (17) we easily obtain the relation

$$\Omega^{+}(t) - \Omega^{-}(t) = \int_t^{\infty} [K^{-}(t,u) - K^{+}(t,u)]du. \tag{19}$$

The final equations (18)–(19) lead us to the following system of equations that helps us determine the relationships $V_{p,\alpha}^{(\pm)}$ and $V_{\alpha}^{(\mp)}$:

$$\begin{gathered} V_\alpha^{+} + V_\alpha^{-} + \textstyle\sum_{s=1}^{\alpha-1} V_s^{+} V_{\alpha-s}^{-} = 0, \\ V_\alpha^{+} - V_\alpha^{-} + \textstyle\sum_{p=1}^{\alpha} \frac{V_{p\alpha}^{+} - V_{p\alpha}^{-}}{p} = 0. \end{gathered} \tag{20}$$

Now let us rewrite (20) as

$$V_{\nu\alpha+\nu}^{(\pm)} = V_{\nu\nu}^{(\pm)}\left(V_{\alpha}^{(\mp)} + \sum_{p=1}^{\alpha} \frac{V_{p\alpha}^{(\mp)}}{\alpha_p + \alpha_\nu}\right), \nu, \alpha = 1, 2, 3, \dots. \tag{21}$$

Let $\tilde{V}^{\pm}_{\nu\alpha+\nu}$, $\nu,\alpha = 1,2,3,\dots$, be a solution of equation (21) when $V^{\pm}_{\alpha} = 1$ and $\hat{V}^{\pm}_{\nu\alpha+\nu}$ is a solution of the same equation corresponding to the case $V^{\pm}_{\alpha} = \pm i$. Then

$$\begin{aligned} \tilde{V}^{\pm}_{\nu\alpha+\nu} &= V^{\pm}_{\nu\nu}\left(1 + \sum_{p=1}^{\alpha} \frac{\tilde{V}^{\mp}_{p\alpha}}{p+\nu}\right), \\ \tilde{V}^{\pm}_{\nu\alpha+\nu} &= V^{\pm}_{\nu\nu}\left(1 + \sum_{p=1}^{\alpha} \frac{\tilde{V}^{\mp}_{p\alpha}}{p+\nu}\right). \end{aligned} \tag{22}$$

Let $\gamma^{\pm}_{\nu\alpha}$ and $\beta^{\mp}_{\nu\alpha}$ be the functions defined as

$$\begin{aligned} \gamma^{\pm}_{\nu\alpha} &= \frac{1}{2}\left[\tilde{V}^{\mp}_{\nu\alpha+\nu} \mp i\hat{V}^{\mp}_{\nu\alpha+\nu}\right], \\ \beta^{\mp}_{\nu\alpha} &= \frac{1}{2}\left[\tilde{V}^{\mp}_{\nu\alpha+\nu} \pm i\hat{V}^{\mp}_{\nu\alpha+\nu}\right]. \end{aligned}$$

Note that the quantities $\gamma^{\pm}_{\nu\alpha}$ and $\beta^{\mp}_{\nu\alpha}$ are determined uniquely using the recurrence relations (22) for the known numbers $V^{\pm}_{\nu\nu}$. Then it is easy to obtain the relation

$$V^{\pm}_{\nu\alpha+\nu} = V^{\mp}_{\alpha}\cdot\gamma^{\pm}_{\nu\alpha} + V^{\pm}_{\alpha}\cdot\beta^{\mp}_{\nu\alpha}. \tag{23}$$

Then using (23) in (20), we have

$$\begin{aligned} &\sum_{p=1}^{\alpha} \frac{V^{+}_{p\alpha}-V^{-}_{p\alpha}}{p} = \sum_{p=1}^{\alpha} \frac{V^{-}_{\alpha}\gamma^{+}_{p\alpha-p}+V^{+}_{\alpha}\beta^{-}_{p\alpha-p}-V^{+}_{\alpha}\gamma^{-}_{p\alpha-p}-V^{-}_{\alpha}\beta^{+}_{p\alpha-p}}{p} = \\ &= V^{-}_{\alpha}\sum_{p=1}^{\alpha}\frac{\gamma^{+}_{p\alpha-p}-\beta^{+}_{p\alpha-p}}{p} + V^{+}_{\alpha}\sum_{p=1}^{\alpha}\frac{\beta^{-}_{p\alpha-p}-\gamma^{-}_{p\alpha-p}}{p}. \end{aligned}$$

Finally, from (20) we obtain that

$$V^{+}_{\alpha}(1-\sum_{p=1}^{\alpha}\frac{\beta^{-}_{p\alpha-p}-\gamma^{-}_{p\alpha-p}}{p}) - V^{-}_{\alpha}(1-\sum_{p=1}^{\alpha}\frac{\gamma^{+}_{p\alpha-p}-\beta^{+}_{p\alpha-p}}{p}) = 0. \tag{24}$$

Let us denote

$$\Lambda_{\alpha} = \frac{1-\sum_{p=1}^{\alpha}\frac{\beta^{-}_{p\alpha-p}-\gamma^{-}_{p\alpha-p}}{n}}{1-\sum_{p=1}^{\alpha}\frac{\gamma^{+}_{p\alpha-p}-\beta^{+}_{p\alpha-p}}{p}}.$$

Then from (24) we obtain

$$V^{+}_{\alpha} = V^{-}_{\alpha}\Lambda_{\alpha} \tag{25}$$

and

$$V^{-}_{\alpha}(1+\Lambda_{\alpha}) + \sum_{s=1}^{\alpha-1} V^{-}_{s}V^{-}_{\alpha-s}\Lambda_{s} = 0. \tag{26}$$

Formulas (25) and (26) uniquely determine all numbers $V^{\pm}_{\alpha}$. Then from (23) all numbers $V^{\pm}_{p\alpha}$ are determined uniquely and effectively. The theorem is proved. □

5 Conclusions

In our investigation, we delve into the implications of complex potentials on the structure of eigenvalues and the nature of spectral singularities. The transfer matrix method, renowned for its efficiency in handling Sturm–Liouville problems, allows us to track the evolution of wave functions through the potential landscape. By establishing a relationship between the transfer matrices and the corresponding complex potentials, we can derive the conditions that lead us to the manifestation of spectral singularities in the eigenvalue spectrum. Furthermore, the presence of almost periodicity in the complex potential introduces a rich tapestry of behaviors not typically observed in real potential scenarios. This aspect opens up avenues for a detailed analysis of the stability of spectral properties, revealing how small perturbations in the potential can lead to significant changes in the eigenvalue distribution. Our findings highlight how eigenvalues cluster or disperse under specific configurations, emphasizing the complex interplay of periodicity and spectral characteristics. Moreover, the implications of our findings extend to practical applications, particularly in the realm of quantum mechanics, where the behavior of wave functions is intricately linked to Sturm–Liouville problems. A precise characterization of eigenvalues and eigenfunctions associated with these differential operators can lead to a better understanding of quantum systems, especially those with nonstandard potential landscapes. This research not only enriches the theoretical framework but also has the potential to influence the development of more sophisticated quantum models. Additionally, the connection between impulsive Sturm–Liouville issues and wave dynamics merits further exploration. By investigating how impulsive effects alter wave propagation researchers can gain insights into phenomena such as solitons and dispersive waves. This could pave the way for novel approaches to controlling wave behavior in various media, which has significant implications in fields such as optics and acoustics. Finally, the incorporation of intricate potentials in our analysis invites further inquiry into their mathematical properties and physical relevance. As we advance our understanding of these complex interactions, we open avenues for interdisciplinary research that could bridge theoretical mathematics and applied physics, ultimately leading to new technologies and methodologies in solving real-world problems [23–42]. Thus our work serves as a stepping stone toward a richer comprehension of spectral theory and its applications across various scientific domains.

Acknowledgements
We truly appreciate the hard work of the editor and reviewers in refining the manuscript. Their significant input and dedication to high standards have substantially enhanced the clarity and quality of our research

Author contributions
S.A., R.F. and D. J., and wrote the main manuscript text and A.M. supervisors and funding of the paper. All authors reviewed the manuscript.

Funding
This work was funded by the Deanship of Scientific Research at King Khalid University through a large group research project under grant number RGP2/58/45.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Baku Engineering University, Baku, AZ0102, Azerbaijan. [2]Scientific Research Center, Department of Mathematics, Baku Engineering University, Baku, AZ0102, Azerbaijan. [3]Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, Uzbekistan. [4]Department of Mathematics, College of Science, King Khalid University, Abha, 61413, Saudi Arabia.

Received: 7 December 2024 Accepted: 25 February 2025 Published online: 04 March 2025

References

1. Choucha, A., Boulaaras, S., Ouchenane, D., Cherif, B., Hidan, M., Abdalla, M.: Exponential stabilization of a swelling porous-elastic system with microtemperature effect and distributed delay. J. Funct. Spaces **2021**, Article ID 5513981 (2021)
2. Yazid, F., Saadallah, A., Ouchenane, D., Chougui, N., Abdalla, M., Himadan, A.: Asymptotic behavior of weak solutions of nonisothermal flow of Herschel-Bulkley fluid to free boundary. Discrete Dyn. Nat. Soc. **2022**, Article ID 5610938 (2022)
3. Ake, M., Hidan, M., Abdalla, M.: Complex boundary value problems for the Cauchy–Riemann operator on a triangle. Fractals **30**(10), Article ID 2240252 (2022)
4. Yang, C.-F.: Inverse problems for the Sturm–Liouville operator with discontinuity. Inverse Probl. Sci. Eng. **22**, 232–244 (2014)
5. Krueger, R.J.: Inverse problems for nonabsorbing media with discontinuous material properties. J. Math. Phys. **23**, 396–404 (1982)
6. Yang, C.F., Bondarenko, N.: Local solvability and stability of inverse problems for Sturm–Liouville operators with a discontinuity. J. Differ. Equ. **268**, 1445–1469 (2020)
7. Yang, C.-F., Guo, Y.: Determination of a differential pencil from interior spectral data. J. Math. Anal. Appl. **375**, 284–293 (2011)
8. Mamedov, K.R., Akcay, O.: Inverse eigenvalue problem for a class of Dirac operators with discontinuous coefficient. Bound. Value Probl. **110**, 1–20 (2014)
9. Bairamov, E., Erdal, I., Yardimci, S.: Spectral properties of an impulsive Sturm–Liouville operator. J. Inequal. Appl. **171**, 1–6 (2016)
10. Hald, O.H.: Discontinuous inverse eigenvalue problems. Commun. Pure Appl. Math. **37**, 539–577 (1984)
11. Gasymov, M.G.: Spectral analysis of a class of second-order non-self-adjoint differential operators. Funct. Anal. Appl. **14**, 11–15 (1980)
12. Efendiev, R.F.: Spectral analysis of a class of non-self-adjoint differential operator pencils with a generalized function. Theor. Math. Phys. **145**(1), 1457–1461 (2005)
13. Efendiev, R.F., Orudzhev, H.D., Bahlulzade, S.J.: Spectral analysis of the discontinuous Sturm–Liouville operator with almost-periodic potentials. Adv. Math. Model. Appl. **6**, 266–277 (2021)
14. Efendiev, R., Gasimov, Y.: Spectral analysis of the indefinite non-self-adjoint Sturm–Liouville operator. Partial Differ. Equ. Appl. Math. **11**(1), 100831 (2024)
15. Efendiev, R., Gasimov, Y.: Inverse spectral problem for PT-symmetric Schrödinger operator on the graph with loop. Glob. Stoch. Anal. **9**(2), 67–77 (2022)
16. Bainov, D.D., Simeonov, P.S.: Impulsive Differential Equations: Asymptotic Properties of the Solutions. Series on Advances in Mathematics for Applied Sciences, vol. 28. World Scientific, Singapore (1995)
17. Perestyuk, N.A., Plotnikov, V.A., Samoilenko, A.M., Skripnik, N.V.: Differential Equations with Impulse Effects: Multivalued Right-Hand Sides with Discontinuities. De Gruyter Studies in Mathematics, vol. 40. de Gruyter, Berlin (2011)
18. Ugurlu, E., Bairamov, E.: Dissipative operators with impulsive conditions. J. Math. Chem. **51**, 1670–1680 (2013)
19. Allahverdiev, B.P., Bairamov, E., Ugurlu, E.: Eigenparameter dependent Sturm–Liouville problems in boundary conditions with transmission conditions. J. Math. Anal. Appl. **401**, 388–396 (2013)
20. Allahverdiev, B.P., Tuna, H.: Eigenfunction expansion for singular Sturm–Liouville problems with transmission conditions. Electron. J. Differ. Equ. **2019**, 3 (2019)
21. Allahverdiev, B.P., Uğurlu, E.: Scattering and spectral problems of the direct sum Sturm–Liouville operators. Appl. Comput. Math. **16**(3), 257–268 (2017)
22. Jaulent, M., Jean, C.: The inverses-wave scattering problem for a class of potentials depending on energy. Commun. Math. Phys. **28**, 177–220 (1972)
23. Orudzhev, A.D.: Spectral analysis of a class of higher-order nonselfadjoint differential operators. Dokl. Akad. Nauk Azerb. SSR **37**(2), 8–11 (1981)
24. Sarnak, P.: Spectral behavior of quasi-periodic potentials. Commun. Math. Phys. **84**(3), 377–401 (1982)
25. Juraev, D.A.: The Cauchy problem for matrix factorizations of the Helmholtz equation in $\mathbb{R}^3$. Adv. Math. Model. Appl. **1**(4), 86–96 (2019)
26. Juraev, D.A.: On the solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional spatial domain. Glob. Stoch. Anal. **9**(2), 1–17 (2022)
27. Juraev, D.A., Noeiaghdam, S.: Modern problems of mathematical physics and their applications. Axioms **11**(2), 1–6 (2022)
28. Juraev, D.A., Shokri, A., Marian, D.: On an approximate solution of the Cauchy problem for systems of equations of elliptic type of the first order. Entropy **24**(7), 1–18 (2022)
29. Juraev, D.A., Shokri, A., Marian, D.: On the approximate solution of the Cauchy problem in a multidimensional unbounded domain. Fractal Fract. **6**(7), 1–14 (2022)
30. Juraev, D.A., Shokri, A., Marian, D.: Solution of the ill-posed Cauchy problem for systems of elliptic type of the first order. Fractal Fract. **6**(7), 1–11 (2022)
31. Juraev, D.A., Shokri, A., Marian, D.: Regularized solution of the Cauchy problem in an unbounded domain. Symmetry **14**(8), 1–16 (2022)

32. Bulnes, J.: An unusual quantum entanglement consistent with Schrödinger's equation. Glob. Stoch. Anal. **9**(2), 78–87 (2022)
33. Bulnes, J.: Solving the heat equation by solving an integro-differential equation. Glob. Stoch. Anal. **9**(2), 89–97 (2022)
34. Zeng, S.D., Bai, Y.R., Winker, P., Yao, J.C.: Identification of discontinuous parameters in double phase obstacle problems. Adv. Nonlinear Anal. **12**(1), 1–22 (2023). https://doi.org/10.1515/anona-2022-0223
35. Abdelwahed, M., Chor, N.: On the topological gradient method for an inverse problem resolution. Adv. Nonlinear Anal. **12**(1), 20230109 (2023). https://doi.org/10.1515/anona-2023-0109
36. Hoffmann-Ostenhof, T., Laptev, A., Shcherbakov, I.: Hardy and Sobolev inequalities on antisymmetric functions. Bull. Math. Sci. **14**(1), 2350010 (2023). https://doi.org/10.1142/S1664360723500108
37. Dudko, A., Lesechko, O., Pivovarchik, V.: Recovering the shape of with the Dirichlet conditions at the pendant vertices. Opusc. Math. **44**(5), 689–705 (2024)
38. Kabatas, A.: One boundary value problem including a spectral parameter in all boundary conditions. Opusc. Math. **43**(5), 651–661 (2023). https://doi.org/10.7494/OpMath.2023.43.5.651
39. Aliyeva, K.H.: On the Construction of the Resolvent of a Differential Pencil of the Fourth Order on the Whole Real Line. J. Contemp. Appl. Math. **14**(2) (2024)
40. Hadiyeva, S.S.: Basis Property of the System of Eigenfunctions Corresponding to a Problem with a Spectral Parameter in the Boundary Condition. J. Contemp. Appl. Math. **14**(22) (2024)
41. Col, A.: Inverse scattering problem for Sturm–Liouville equation with a rational function of spectral parameter in the boundary condition. J. Contemp. Appl. Math. **13**(1) (2024)
42. Niyozov, E., Juraev, D.A., Efendiev, R.F., Jalalov, M.J.: Periodic solutions of second order impulsive system for integrodifferential equations with maxima. J. Contemp. Appl. Math. **14**(22) (2024)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com