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MULTIPLICITY OF EIGENVALUES OF THE DIFFUSION OPERATOR WITH A SPECTRAL PARAMETER QUADRATICALLY CONTAINED IN THE BOUNDARY CONDITION

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ARTICLE INFO	ABSTRACT
Article history: Received:2024-10-08 Received in revised form:2024-10-08 Accepted:2024-11-19 Available online *Keywords: diffusion operator; nonseparated boundary conditions; eigenvalues; spectral parameter.*	*The paper considers a boundary value problem generated by a differential diffusion equation and nonseparated boundary conditions. One of the boundary conditions contains a quadratic function of the spectral parameter. The multiplicity of eigenvalues of the boundary value problem under consideration is investigated. A criterion for the multiplicity of eigenvalues and zeros of the characteristic function of a boundary value problem is obtained.The found necessary and sufficient conditions are expressed through the values of the fundamental solutions of the diffusion equation and the coefficients of the boundary conditions. Note that the results obtained can be used in the study of direct and inverse problems of spectral analysis for various differential operators. These results also play an important role in studying the structure of the spectrum, in establishing the order of arrangement of eigenvalues of boundary value problems, and in finding sufficient conditions for the reconstruction of the corresponding problems.*

Let's consider a boundary value problem defined on an interval $[0,\pi]$ by the diffusion differential equation

$$y'' + [\lambda^2 - 2\lambda p(x) - q(x)]y = 0 \tag{1}$$

and boundary conditions of the form

$$\begin{aligned} &(m\lambda^2 + \alpha\lambda + \beta)y(0) + y'(0) + \omega y(\pi) = 0, \\ &-\overline{\omega} y(0) + \gamma y(\pi) + y'(\pi) = 0, \end{aligned} \tag{2}$$

where $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real functions, λ is the spectral parameter, ω is complex, $\overline{\omega}$ is the conjugate to ω, and m, α, β, γ are real numbers.

Using $W_2^n[0,\pi]$, we denote the Sobolev space of functions $f(x)$, $x \in [0,\pi]$, such that functions $f^{(m)}(x)$, $m = 0,1,2,\ldots,n-1$, are absolutely continuous and $f^{(n)}(x) \in L_2[0,\pi]$. We will denote problem (1)-(2) by P.

Definition. The values of the parameter λ for which the boundary value problem P has nontrivial solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions of problem P. The set of eigenvalues is called the spectrum of P. The number of linearly independent solutions of problem P for a given eigenvalue λ_0 is called the multiplicity of the eigenvalue λ_0.

The spectral properties of Sturm-Liouville and diffusion operators with separated boundary conditions (i.e., at $\omega = 0$) are well studied (see works [1-4] and the bibliography within them). It has been proven that, in this case, the eigenvalues of boundary value problems are simple (i.e., each eigenvalue corresponds to a unique eigenfunction, up to a constant multiplier). In [5-18], direct and inverse spectral problems for the equation (1) (under conditions $p(x) \equiv 0$ and $p(x) \neq 0$) with various types of non-separated boundary conditions have been investigated, including periodic, anti-periodic, quasi-periodic, and generalized periodic boundary conditions.In the article [19], a criterion for multiplicity was obtained, and the order of the eigenvalue distribution of the quadratic pencil of Sturm-Liouville operators was established in the case of non-separated boundary conditions without a spectral parameter. The multiplicity of the eigenvalues of diffusion and Dirac operators under non-separated boundary conditions containing a linear function of the spectral parameter has been studied in [20-21].

In this work, necessary and sufficient conditions are found for the multiplicity of eigenvalues (as well as for the multiplicity of the zero of the characteristic function) of the boundary value problem P.

Further, we will assume that $m > 0$ and that, for all functions $y(x) \in W_2^2[0, \pi]$, $y(x) \not\equiv 0$ satisfying conditions (2), the inequality

$$\gamma |y(\pi)|^2 - 2\,\mathrm{Re}\left[\omega \overline{y(0)} y(\pi)\right] - \beta |y(0)|^2 + \int_0^\pi \left\{ |y'(x)|^2 + q(x)|y(x)|^2 \right\} dx > 0$$

holds.

Under this condition, the eigenvalues of the boundary value problem P are real and non-zero, and this problem has no associated functions corresponding to the eigenfunctions. Furthermore, if $y(x)$ is an eigenfunction of problem P corresponding to the eigenvalue λ, then

$$2\int_0^\pi [\lambda - p(x)] |y(x)|^2 dx + (2m\lambda + \alpha)|y(0)|^2 \neq 0. \tag{3}$$

(see [15]).

Let's introduce the fundamental system of solutions $c(x,\lambda)$, $s(x,\lambda)$ of equation (1), defined by the initial conditions

$$c(0,\lambda) = s'(0,\lambda) = 1,\ c'(0,\lambda) = s(0,\lambda) = 0. \tag{4}$$

The Wronskian of these solutions is identically equal to one:

$$\begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ c'(x,\lambda) & s'(x,\lambda) \end{vmatrix} = c(x,\lambda)s'(x,\lambda) - s(x,\lambda)c'(x,\lambda) = 1 \tag{5}$$

Let us denote

$$a(\lambda) = m\lambda^2 + \alpha\lambda + \beta. \tag{6}$$

It is clear that the eigenvalues of boundary value problem P coincide with the zeros of the function

$$\Delta(\lambda) = \begin{vmatrix} a(\lambda) + \omega c(\pi,\lambda) & 1 + \omega s(\pi,\lambda) \\ -\overline{\omega} + \gamma c(\pi,\lambda) + c'(\pi,\lambda) & \gamma s(\pi,\lambda) + s'(\pi,\lambda) \end{vmatrix}.$$

This function is called the characteristic function of problem P. Expanding this determinant and taking into account identity (5), we have

$$\Delta(\lambda) = 2\operatorname{Re}\omega - \eta(\lambda) + |\omega|^2 s(\pi,\lambda) + a(\lambda)\sigma(\lambda), \tag{7}$$

where

$$\eta(\lambda) = c'(\pi,\lambda) + \gamma\, c(\pi,\lambda),\ \ \sigma(\lambda) = s'(\pi,\lambda) + \gamma\, s(\pi,\lambda)\ . \tag{8}$$

Theorem 1. *Eigenvalue λ_0 of boundary value problem P will be multiple if and only if the number is real and nonzero and the following equalities are satisfied:*

$$a(\lambda_0) + \omega c(\pi,\lambda_0) = \sigma(\lambda_0) = 0\,. \tag{9}$$

The proof of this theorem is similar to the proof of Theorem 2.1 in [19].

The eigenvalue λ_0 can be a multiple of zero of the characteristic function $\Delta(\lambda)$ of boundary value problem P.

Theorem 2. *In order for the zero λ_0 of the characteristic function $\Delta(\lambda)$ of the boundary value problem P to be multiple, it is necessary and sufficient that the conditions for the multiplicity of the eigenvalue λ_0 be satisfied, i.e. the number ω is real and nonzero and equalities (9) are valid.*

Proof. *Necessity.* Hereafter, we will denote the derivative with respect to parameter λ by a dot over the function. First, we prove that if $\Delta(\lambda_0) = 0$ and $\sigma(\lambda_0) \neq 0$, then $\dot{\Delta}(\lambda_0) \neq 0$.
According to (1) and (4), we have

$$\dot{c}''(x,\lambda) + [\lambda^2 - 2\lambda p(x) - q(x)]\dot{c}'(x,\lambda) = 2[p(x) - \lambda]c(x,\lambda)$$
$$\dot{c}(0,\lambda) = \dot{c}'(0,\lambda) = 0$$

As in [19] we find

$$\dot{c}(x,\lambda) = \int_0^x h(x,t,\lambda)c(t,\lambda)dt\,,\ \ \dot{s}(x,\lambda) = \int_0^x h(x,t,\lambda)s(t,\lambda)dt\ , \tag{10}$$

$$\dot{c}'(x,\lambda) = \int_0^x h'_x(x,t,\lambda)c(t,\lambda)dt\,,\ \dot{s}'(x,\lambda) = \int_0^x h'_x(x,t,\lambda)s(t,\lambda)dt\ , \tag{11}$$

where

$$h(x,t,\lambda) = 2\big(c(x,\lambda)s(t,\lambda) - s(x,\lambda)c(t,\lambda)\big)\cdot\big(\lambda - p(t)\big), \tag{12}$$

$$h'_x(x,t,\lambda) = 2\big(c'(x,\lambda)s(t,\lambda) - s'(x,\lambda)c(t,\lambda)\big)\cdot\big(\lambda - p(t)\big). \tag{13}$$

To simplify the notation, in the future we will often write $c, c', s, s'.$ instead of $c(\pi,\lambda), c'(\pi,\lambda), s(\pi,\lambda), s'(\pi,\lambda)$.

Let's find the derivative of function (7) with respect to λ:

$$\dot{\Delta}(\lambda) = |\omega|^2 \dot{s} + (2m\lambda + \alpha)\sigma(\lambda) + a(\lambda)\dot{\sigma}(\lambda) - \dot{\eta}(\lambda). \tag{14}$$

Taking into account (6), (10) and (11) in the last equality, we obtain

$$\dot{\Delta}(\lambda) = \int_0^\pi |\omega|^2 h(\pi,t,\lambda)s(t,\lambda)dt + a(\lambda)\cdot\left[\int_0^\pi h'_x(\pi,t,\lambda)s(t,\lambda)dt + \int_0^\pi \gamma h(\pi,t,\lambda)s(t,\lambda)dt\right] -$$

$$-\int_0^\pi \gamma h(\pi,t,\lambda)c(t,\lambda)dt - \int_0^\pi h'_x(\pi,t,\lambda)c(t,\lambda)dt + (2m\lambda+\alpha)\sigma(\lambda).$$

From here, by virtue of (8), (12) and (13), it follows

$$\dot{\Delta}(\lambda) = 2\int_0^\pi \Big[\sigma(\lambda)c^2(\lambda,t) - \big(\eta(\lambda) + |\omega|^2 s + a(\lambda)\sigma(\lambda)\big)c(\lambda,t)s(\lambda,t) +$$

$$+\big(|\omega|^2 c + a(\lambda)\eta(\lambda)\big)s^2(\lambda,t)\Big]\big(\lambda - p(t)\big)dt + (2m\lambda+\alpha)\sigma(\lambda). \tag{15}$$

Since $\Delta(\lambda_0) = 0$, then λ_0 is an eigenvalue of problem P. Let $f_0(x)$ be the corresponding eigenfunction. It is easy to verify that

$$f_0(x) = f_0(0)c(x,\lambda_0) + f_0'(0)s(x,\lambda_0). \tag{16}$$

By virtue of this equality we obtain

$$|f_0(x)|^2 = f_0(x)\overline{f_0(x)} = |f_0(0)|^2 c^2(\lambda_0,x) + |f_0'(0)|^2 s^2(\lambda_0,x) +$$

$$+\Big[f_0(0)\overline{f_0'(0)} + \overline{f_0(0)}f_0'(0)\Big]c(\lambda_0,x)s(\lambda_0,x). \tag{17}$$

From (16) we obtain

$$f_0'(\pi) = f_0(0)c'(\pi,\lambda_0) + f_0'(0)s'(\pi,\lambda_0).$$

According to the second of the boundary conditions (2) (taking into account (17)) we have

$$-\overline{\omega}f_0(0) + \gamma f_0(0)c(\pi,\lambda_0) + \gamma f_0'(0)s(\pi,\lambda_0) + f_0(0)c'(\pi,\lambda_0) + f_0'(0)s'(\pi,\lambda_0) = 0$$

or

$$f_0(0)(-\overline{\omega} + \eta(\lambda_0)) + f_0'(0)\sigma(\lambda_0) = 0$$

From here

$$f_0'(0)=\frac{\bar{\omega}-\eta(\lambda_0)}{\sigma(\lambda_0)}\,f_0(0). \tag{18}$$

It is obvious that

$$|f_0'(0)|^2 = f_0'(0)\overline{f_0'(0)}=\frac{\bar{\omega}-\eta(\lambda_0)}{\sigma(\lambda_0)}\cdot\frac{\omega-\eta(\lambda_0)}{\sigma(\lambda_0)}|f_0(0)|^2 =$$

$$=\frac{|\omega|^2-\bar{\omega}\eta(\lambda_0)-\omega\eta(\lambda_0)+\eta^2(\lambda_0)}{\sigma^2(\lambda_0)}|f_0(0)|^2 =$$

$$=\frac{|\omega|^2-\eta(\lambda_0)(2\operatorname{Re}\omega-\eta(\lambda_0))}{\sigma^2(\lambda_0)}|f_0(0)|^2.$$

Hence,

$$|f_0'(0)|^2=\frac{|\omega|^2+\eta(\lambda_0)\lfloor a(\lambda_0)\sigma(\lambda_0)+|\omega|^2 s)\rfloor}{\sigma^2(\lambda_0)}|f_0(0)|^2. \tag{19}$$

Using identity (5) and relations (18) and (19), we have

$$|f_0'(0)|^2=\frac{|\omega|^2+(c'+\gamma c)\lfloor a(\lambda_0)\sigma(\lambda_0)+|\omega|^2 s\rfloor}{\sigma^2(\lambda_0)}|f_0(0)|^2 =$$

$$=\frac{|\omega|^2+a(\lambda_0)\sigma(\lambda_0)c'+|\omega|^2(cs'-1)+a(\lambda_0)\gamma cs'+a(\lambda_0)\gamma^2 cs+\gamma|\omega|^2 cs}{\sigma^2(\lambda_0)}|f_0(0)|^2 =$$

$$=\frac{a(\lambda_0)\sigma(\lambda_0)c'+|\omega|^2 c\sigma(\lambda_0)+a(\lambda_0)\sigma(\lambda_0)\gamma c}{\sigma^2(\lambda_0)}|f_0(0)|^2 =$$

$$=\frac{a(\lambda_0)\eta(\lambda_0)+|\omega|^2 c}{\sigma(\lambda_0)}|f_0(0)|^2,$$

$$f_0(0)\overline{f_0'(0)}+\overline{f_0(0)}f_0'(0)=\frac{\omega-\eta(\lambda_0)}{\sigma(\lambda_0)}\overline{f_0(0)}f_0(0)+\frac{\bar{\omega}-\eta(\lambda_0)}{\sigma(\lambda_0)}\overline{f_0(0)}f_0(0)=$$

$$=\frac{2\operatorname{Re}\omega-2\eta(\lambda_0)}{\sigma(\lambda_0)}|f_0(0)|^2.$$

According to the ratio

$$\Delta(\lambda_0)=2\operatorname{Re}\omega+|\omega|^2 s+a(\lambda_0)\sigma(\lambda_0)-\eta(\lambda_0)=0 \tag{20}$$

we have

$$\frac{2\operatorname{Re}\omega-2\eta(\lambda_0)}{\sigma(\lambda_0)}|f_0(0)|^2=-\frac{|\omega|^2 s+a(\lambda_0)\sigma(\lambda_0)+\eta(\lambda_0)}{\sigma(\lambda_0)}|f_0(0)|^2$$

Therefore, from (17) we obtain that

$$|f_0(x)|^2 = |f_0(0)|^2 c^2(\lambda_0, x) + \frac{a(\lambda_0)c' + |\omega|^2 c + a(\lambda_0)\gamma c}{\sigma(\lambda_0)} s^2(\lambda_0, x)|f_0(0)|^2 +$$

$$+\frac{-c' - \gamma c - |\omega|^2 s - a(\lambda_0)\gamma s - a(\lambda_0)s'}{\sigma(\lambda_0)} s(\lambda_0, x)c(\lambda_0, x)|f_0(0)|^2 =$$

$$= \left[\frac{\left[|\omega|^2 c + a(\lambda_0)\eta(\lambda_0)\right]s^2(\lambda_0, x) + \left[-|\omega|^2 s - a(\lambda_0)\sigma(\lambda_0) - \eta(\lambda_0)\right]s(\lambda_0, x)c(\lambda_0, x)}{\sigma(\lambda_0)} + \right.$$

$$\left. + \frac{\sigma(\lambda_0)c^2(\lambda_0, x)}{\sigma(\lambda_0)}\right]|f_0(0)|^2.$$

Multiplying both sides of this equality by $\lambda_0 - p(x)$, then integrating within $[0, \pi]$ and taking into account (15) and (17), we get

$$2\int_0^\pi [\lambda_0 - p(x)]|f_0(x)|^2 = \frac{|f_0(0)|^2}{\sigma(\lambda_0)}\left[\dot{\Delta}(\lambda_0) - (2m\lambda_0 + \alpha)\sigma(\lambda_0)\right].$$

Thus,

$$2\int_0^\pi [\lambda_0 - p(x)]|f_0(x)|^2 + (2m\lambda_0 + \alpha)|f_0(0)|^2 = \frac{|f_0(0)|^2}{\sigma(\lambda_0)}\dot{\Delta}(\lambda_0).$$

From here, by virtue of (3), we have $\dot{\Delta}(\lambda_0) \neq 0$.

Let us assume that $\Delta(\lambda_0) = \dot{\Delta}(\lambda_0) = 0$. If $\sigma(\lambda_0) \neq 0$, then, according to what was proved above, $\dot{\Delta}(\lambda_0) \neq 0$ takes place, which contradicts our condition. Hence $\sigma(\lambda_0) = 0$. Then, by virtue of identity

$$c(\pi, \lambda_0)\sigma(\lambda_0) - s(\pi, \lambda_0)\eta(\lambda_0) = 1 \tag{21}$$

we obtain

$$s(\pi, \lambda_0)\eta(\lambda_0) = -1. \tag{22}$$

From $\Delta(\lambda_0) = 0$ according to (7) we have

$$2\operatorname{Re}\omega + |\omega|^2 s(\pi, \lambda_0) + \frac{1}{s(\pi, \lambda_0)} = 0. \tag{23}$$

From here, taking into account the relation

$$|\omega|^2 = \operatorname{Re}^2\omega + \operatorname{Im}^2\omega,$$

we obtain

$$1 + 2s(\pi, \lambda_0)\operatorname{Re}\omega + s^2(\pi, \lambda_0)\operatorname{Re}^2\omega + s^2(\pi, \lambda_0)\operatorname{Im}^2\omega = 0,$$
$$(1 + s(\pi, \lambda_0)\operatorname{Re}\omega)^2 + (s(\pi, \lambda_0)\operatorname{Im}\omega)^2 = 0,$$

i.e.

$$1+s(\pi,\lambda_0)\operatorname{Re}\omega = s(\pi,\lambda_0)\operatorname{Im}\omega = 0. \tag{24}$$

By virtue of (22) we have $s(\pi,\lambda_0)\neq 0$. Then from (24) it follows that

$$\operatorname{Im}\omega = 0,\ \omega\neq 0,\quad 1+\omega s(\pi,\lambda_0)=0.$$

It remains to prove that the equality

$$a(\lambda_0)+\omega c(\pi,\lambda_0)=0$$

also holds.

Since ω is real, then $\operatorname{Re}\omega=\omega,\ |\omega|^2=\omega^2$. Then from (23)

$$-2\omega-\omega^2 s(\pi,\lambda_0)+\eta(\lambda_0)=0.$$

Since $\omega s(\pi,\lambda_0)=-1$, then

$$2\omega^2 s(\pi,\lambda_0)-\omega^2 s(\pi,\lambda_0)+\eta(\lambda_0)=0,$$

or

$$\omega^2 s(\pi,\lambda_0)+\eta(\lambda_0)=0. \tag{25}$$

Therefore, from formula (15) it follows that

$$\left[\omega^2 c+a(\lambda_0)\eta(\lambda_0)\right]\int_0^\pi \left[\lambda_0-p(t)\right]s^2(t,\lambda_0)dt=0. \tag{26}$$

Since $1+\omega s(\pi.\lambda_0)=\sigma(\lambda_0)=0$ and $\operatorname{Im}\omega=0$, then

$$a(\lambda_0)s(0,\lambda_0)+s'(0,\lambda_0)+\omega s(\pi,\lambda_0)=0,$$

$$-\omega s(0,\lambda_0)+\gamma s(\pi,\lambda_0)+s'(\pi,\lambda_0)=0.$$

This means that function $s(x,\lambda_0)$ satisfies the boundary conditions (2). Then $s(x,\lambda_0)$ is an eigenfunction of the boundary value problem P. Therefore, according to (3)

$$\int_0^\pi \left[\lambda_0-p(t)\right]s^2(t,\lambda_0)dt+(2m\lambda_0+\alpha)|s(0,\lambda_0)|^2\neq 0.$$

From (26) it follows that

$$\omega^2 c(\pi,\lambda_0)+a(\lambda_0)\eta(\lambda_0)=0. \tag{27}$$

Next, by virtue of equality $\omega s(\pi,\lambda_0)=-1$, we sequentially find

$$\eta(\lambda_0)+\omega^2 s(\pi,\lambda_0)=0,\ \eta(\lambda_0)+\omega\cdot\omega s(\pi,\lambda_0)=0,\ \eta(\lambda_0)-\omega=0,\ \eta(\lambda_0)=\omega.$$

Then from (27) we find that $\omega^2 c(\pi,\lambda_0)+a(\lambda_0)\omega=0$. Hence $a(\lambda_0)+\omega c(\pi,\lambda_0)=0$.

Sufficiency. Let $\Delta(\lambda_0)=0$, $\operatorname{Im}\omega=0$, $\omega\neq 0$ and the equalities (9) hold.

According to (20) and (21)

$$\Delta(\lambda_0) = 2\omega - \eta(\lambda_0) + \omega^2 s(\pi, \lambda_0) = 2\omega + \frac{1}{s(\pi, \lambda_0)} + \omega^2 s(\pi, \lambda_0) = 0$$

From here

$$2\omega s(\pi, \lambda_0) + 1 + \omega^2 s^2(\pi, \lambda_0) = 0,$$

$$[\omega s(\pi, \lambda_0) + 1]^2 = 0, \text{ т.е. } \omega s(\pi, \lambda_0) = -1.$$

Then

$$\Delta(\lambda_0) = 2\omega - \eta(\lambda_0) - \omega = \omega - \eta(\lambda_0) = 0,$$

i.e. $\omega = \eta(\lambda_0)$. By the condition $a(\lambda_0) + \omega c(\pi, \lambda_0) = 0$. Then

$$\omega^2 c(\pi, \lambda_0) + a(\lambda_0)\omega = \omega^2 c(\pi, \lambda_0) + a(\lambda_0)\eta(\lambda_0) = 0.$$

Using this equality and taking into account the relations

$$\sigma(\lambda_0) = \eta(\lambda_0) + \omega^2 s(\pi, \lambda_0) + a\sigma(\lambda_0) = 0$$

from (15) we obtain that $\dot{\Delta}(\lambda_0) = 0$, i.e. λ_0 is a multiple zero of the function $\Delta(\lambda)$.

The theorem is proven.

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