INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

Mathematical Modeling and Experimental Research of Microwave Waveguide Taking into Account Boundary Value Problems on the Geometric Middle of the Domain

Islam J. Islamov[1*], Ilgar G. Mamedov[2], Aynura J. Abdullayeva[3]

[1]Department of Automation, Telecommunications and Energy, Baku Engineering University, Hasan Aliyev str., 120, AZ0101, Baku, Azerbaijan
isislamov@beu.edu.az
[2]Institute of Control Systems, Bakhtiyar Vahabzade str., 68, AZ 1141, Baku, Azerbaijan
Sumgait State University, 43rd district, Baku str., 1, AZ5008, Sumgait, Azerbaijan
ilgar-mamedov-1971@mail.ru
[3]Military-Scientific Research Institute, National Defense University, Shafayat Mehdiyev str., 136, AZ1065, Baku, Azerbaijan
aynure-huseynova-2015@mail.ru

Abstract - **The article presents mathematical modeling and experimental study of microwave waveguide taking into account boundary value problems on the geometric middle of the domain. For the first time, general classes of local and nonlocal boundary value problems on the geometric middle of the domain for three-dimensional Bianchi-type equations for the electromagnetic field of microwave rectangular waveguide are investigated. New mathematical models of the electromagnetic field of microwave rectangular waveguide operating in the frequency range of 4,9-7,05 *GHz* are developed taking into account the nonlinearity of the medium, wave types and effective algorithms for solving the models are proposed, which made it possible to improve the electrical, magnetic, structural and operational parameters and characteristics of microwave rectangular waveguide. For *E*-type and *H*-type waves, 3*D* models of the distribution of electromagnetic field strengths in elementary regions of rectangular waveguide are developed. Experimental devices and functional circuits for measuring the parameters of the microwave path, including the electric and magnetic fields of a rectangular waveguide in a nonlinear state of the medium, were developed, and the parameters of the microwave rectangular waveguide were experimentally determined. Comparison of theoretical and experimental results of the electric and magnetic field strengths showed that the relative error for these parameters is 4%. This proved the adequacy of the theoretical and experimental results obtained. The proposed work differs from the existing [1-5] works in that, in this work, for the first time, the factor that the studied microwave waveguide is filled with nonlinear media is taken into account.**

Index Terms - **Microwave Rectangular Waveguide, Electromagnetic Field, Mathematical Modeling, Experimental Research, 3*D* Bianchi Type Equations, Correct Solvability.**

I. INTRODUCTION

At present, for mathematical modeling, it is relevant to study the interactions of electromagnetic waves with complexly structured materials: composite [1-5]; anisotropic, biisotropic, and bianisotropic [6-10]. Of practical interest is the study of the electrodynamic properties of anisotropic media, including crystals, ferrites, magnetized plasma, and gyrotropic materials [11-15]. Single-layer and multilayer screens made of anisotropic and bianisotropic materials are studied [16-20]. Waveguides and waveguide structures are

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

widely used in technical devices of radio engineering [21-26]. In particular, waveguides are used to study the properties of materials that are included in the waveguide in the form of a diaphragm. Also, waveguides with partitions and inclusions are used as filters and wave type converters [27-31]. As a rule, waveguides with perfectly conducting walls are studied, which facilitates analytical modeling. To build more accurate models, it is necessary to take into account the absorption of energy by the walls of waveguides, which are made of various materials. The classical method for modeling absorption in a wall is the Bianchi method of impedance boundary conditions [32-35]. This method has been developed in relation to various types of electromagnetic fields and materials of massive surfaces. Impedance boundary conditions were obtained for anisotropic surfaces [36], for surfaces with multilayer coatings [37, 38], for curved surfaces with allowance for curvature [39] in the case of monochromatic fields, and integral boundary conditions for nonstationary and pulsed fields [40].

In this article, impedance boundary conditions are formulated for scalar potentials on the waveguide wall, taking into account the curvature of the surface under the influence of monochromatic fields from a bianisotropic partition. The solution of the problem of reflection and transmission of symmetric *E*- and *H*-partial waves of a circular waveguide through a bianisotropic-gyrotropic partition is reduced to a numerical solution by the direct grid method of block matrix sweep [39] of a boundary value problem for a system of two elliptic equations with integral radiation boundary conditions [40, 41] at the ends waveguide, taking into account the impedance boundary conditions.

The proposed work differs from the existing [1-5] works in that, in this work for the first time the factor is taken into account that the investigated microwave waveguide is filled with nonlinear media. For the nonlinearity of the microwave rectangular waveguide medium, mathematical models of the electromagnetic field are developed for the first time taking into account boundary value problems on the geometric middle of the region. For the first time, general classes of local and nonlocal boundary value problems on the geometric middle of the region for three-dimensional equations of the Bianchi type for the electromagnetic field of the microwave rectangular waveguide are investigated.

II. DEVELOPMENT OF MATHEMATICAL MODELS OF MICROWAVE RECTANGULAR WAVEGUIDE TAKING INTO ACCOUNT THE STRUCTURAL PROPERTIES OF SPACE

As can be seen from Figure 1, the coordinate origin coincides with one vertex of the rectangle. Let's accept the following notations: a - the size of the wide wall of the rectangular waveguide, b - the size of the narrow wall of the rectangular waveguide, R - the radius of the slits, h - the distance between the centers of the slits.

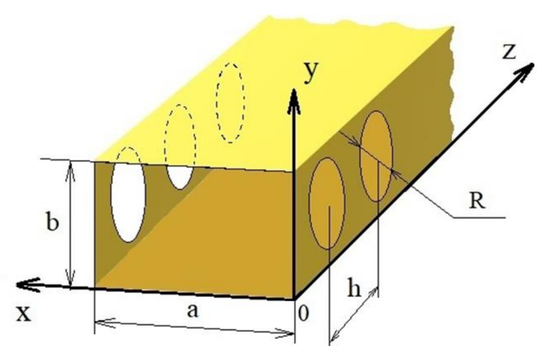

Fig. 1. A rectangular waveguide with a circular slot on the sidewall.

By now, due to the efforts many mathematicians, the theory differential equations with constant or sufficiently smooth coefficients has been developed quite well. In these and other works, various methods have been developed for studying the questions of the correct solvability initial, initial-boundary value problems, as well as questions constructing fundamental solutions for such equations. It should be especially noted that equations with a dominant mixed derivative $E(H)_{xyz}(x, y, z)$ are used in modeling vibration processes and play an essential role in approximation and mapping theories [42-45].

IJMOT

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

Therefore, a very topical question arises about the study questions of correct solvability connected with hyperbolic equations with dominant derivatives, generally speaking, with nonsmooth variable coefficients. In this regard, this demonstrated work is devoted to the study initial-boundary value problems of a new type for hyperbolic equations with dominating mixed derivatives, which, in general, have nonsmooth Lp-coefficients (i.e., coefficients from Lp-type spaces). It is presented mainly in relation to third-order hyperbolic equations with triple characteristics. In this case, an important fundamental point is that the equation under consideration has discontinuous coefficients that satisfy only certain conditions such as P-integrability and boundedness, i.e., the considered hyperbolic differential operator does not have a traditional adjoint operator. In this article, for the study such problems, a technique has been developed that essentially uses modern methods function theory and functional analysis. With the help this technique, a new concept adjoint problem is introduced for initial-boundary value problems. Such adjoint problems, in contrast to traditional adjoint problems defined by formally conjugate differential operators, by definition have the form an integral equation, and therefore make sense under rather weak conditions on the coefficients.

The paper [45-48] study boundary value problems stated in the geometric middle of the domain for $3D$ Bianchi integro-differential equations with nonsmooth coefficients. Here $E(H)=E(H)(x,y,z)$ is a sought for function determined on G:

$$G = G_1 \times G_2 \times G_3, \qquad (1)$$

where $G_1=(x_0,x_1),\ x_0 \ge 0;\ \ G_2=(y_0,y_1),\ y_0 \ge 0;\ \ G_3=(z_0,z_1),\ z_0 \ge 0.$

In these papers, ubder the imposed conditions, the solution $E(H)(x,y,z)$ of the $3D$ Bianchi equation is in the S.L. Sobolev space

$$W_p^{(1,1,1)}(G)=\left\{\begin{array}{l} E(H)\in L_p(G)/D_x^i D_y^j D_z^k u \in L_p(G) \\ i=\overline{0,1},\ j=\overline{0,1},\ k=\overline{0,1} \end{array}\right\} \qquad (2)$$

where $1 \le p \le \infty$.

We will define the norm in the space $W_p^{(1,1,1)}(G)$ by the quality

$$\|E(H)\|_{W_p^{(1,1,1)}(G)} = \sum_{i,j,k=0}^{1} \left\| D_x^i D_y^j D_z^k u \right\|_{L_p(G)}. \qquad (3)$$

At first we note that integral representations of functions from spaces type $W_p^{(1,1,1)}(G)$ (from Sobolev spaces with dominant mixed derivatives of general form) have been studied in the papers of R. Caroca, I. Kondrashuk, N. Merino and F. Nadal [41], A.Y. Kamenshchik [42], S. Azami, M. Jafari, A. Haseeb, A.A.H. Ahmadini [43], F. Ma, Y. Cao, H. Du, et al. [44] and others. But we will use the integral representation from [41], according to which any function $E(H)(x,y,z)\in W_p^{(1,1,1)}(G)$ is uniquely represented in the form

$$\begin{aligned} E(H)(x,y,z) = (Qb)(x,y,z) \equiv b_{0,0,0} + \int_{\sqrt{x_0x_1}}^{x} b_{1,0,0}(\alpha)d\alpha + \int_{\sqrt{y_0y_1}}^{y} b_{0,1,0}(\beta)d\beta + \qquad (4) \\ + \int_{\sqrt{z_0z_1}}^{z} b_{0,0,1}(\gamma)d\gamma + \int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y} b_{1,1,0}(\alpha,\beta)d\alpha d\beta + \int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} b_{0,1,1}(\beta,\gamma)d\beta d\gamma + \\ + \int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{z_0z_1}}^{z} b_{1,0,1}(\alpha,\gamma)d\alpha d\gamma + \int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} b_{1,1,1}(\alpha,\beta,\gamma)d\alpha d\beta d\gamma, \end{aligned}$$

by means of the only element

$$\begin{aligned} b=(b_{0,0,0},b_{1,0,0},b_{0,1,0},b_{0,0,1},b_{1,1,0},b_{0,1,1},b_{1,0,1},b_{1,1,1}) \in \mathrm{E}_p^{(1,1,1)} \equiv \\ \equiv R \times L_p(x_0,x_1) \times L_p(y_0,y_1) \times L_p(z_0,z_1) \times L_p(G_1\times G_2) \times \\ \times L_p(G_2\times G_3) \times L_p(G_1\times G_3) \times L_p(G). \end{aligned}$$

This time, there exist such positive constants M_1^0 and M_2^0 that

$$M_1^0 \|b\|_{\mathrm{E}_p^{(1,1,1)}} \le \|Qb)(x,y,z)\|_{W_p^{(1,1,1)}(G)} \le M_2^0 \|b\|_{\mathrm{E}_p^{(1,1,1)}}, \qquad (5)$$

for all $b \in \mathrm{E}_p^{(1,1,1)}$.

Obviously, the operator $Q:\mathrm{E}_p^{(1,1,1)} \to W_p^{(1,1,1)}(G)$ is a linear bounded operator. The inequality (5) shows that the operator Q has also a bounded inverse operator determined on the space $W_p^{(1,1,1)}(G)$. Consequently,the operator Q is an homeomorphism between the Banach spaces $\mathrm{E}_p^{(1,1,1)}$ and $W_p^{(1,1,1)}(G)$. Furthermore formula (4) shows that any function $E(H)(x,y,z)\in W_p^{(1,1,1)}(G)$ has the traces:

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

$u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}), u_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1}), u_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1}),$
$u_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z), u_{xy}(x,y,\sqrt{z_0z_1}), u_{yz}(\sqrt{x_0x_1},y,z), u_{xz}(x,\sqrt{y_0y_1},z)$
and the operator of taking these traces are continuous from $W_p^{(1,1,1)}(G)$ to R, $L_p(x_0,x_1), L_p(y_0,y_1), L_p(z_0,z_1), L_p(G_1\times G_2), L_p(G_2\times G_3), L_p(G_1\times G_3)$ respectively. Then following equalities are also valid for these traces [42-46]:

$$\begin{gathered} E(H)(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}) = b_{0,0,0}, \\ E(H)_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1}) = b_{1,0,0}(x), \\ E(H)_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1}) = b_{0,1,0}(y), \\ E(H)_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z) = b_{0,0,1}(z), \\ E(H)_{xy}(x,y,\sqrt{z_0z_1}) = b_{1,1,0}(x,y), \\ E(H)_{yz}(\sqrt{x_0x_1},y,z) = b_{0,1,1}(y,z), \\ E(H)_{xz}(x,\sqrt{y_0y_1},z) = b_{1,0,1}(x,z). \end{gathered} \tag{6}$$

III. SOME GENERAL CLASSES WELL-POSED BOUNDARY VALUE PROBLEMS FOR 3*D* BIANCHI TYPE EQUATIONS

The conditions well-defined solvability of the boundary value problem found in [42], show that the operatin taking the traces

$$\begin{gathered} E(H)(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}), \\ E(H)_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1}), \\ E(H)_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1}), \\ E(H)_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z), \\ E(H)_{xy}(x,y,\sqrt{z_0z_1}), \\ E(H)_{yz}(\sqrt{x_0x_1},y,z), \\ E(H)_{xz}(x,\sqrt{y_0y_1},z), \end{gathered} \tag{7}$$

in some sense can be considered as principal parts of the operators

$$V_{0,0,0}, V_{1,0,0}, V_{0,1,0}, V_{0,0,1}, V_{1,1,0}, V_{0,1,1}, V_{1,0,1}$$

respectively generated by boundary conditions. Therefore, it is natural to expect that the problem of finding the solution $E(H)(x,y,z)\in W_p^{(1,1,1)}(G)$ of the boundary value problem of the general form

$$\begin{gathered} (V_{1,1,1}E(H))(x,y,z) = E(H)_{xyz}(x,y,z) + (\hat{V}_{1,1,1}u)(x,y,z) = \\ = \varphi_{1,1,1}(x,y,z),\ (x,y,z)\in G, \end{gathered} \tag{8}$$

$$\begin{cases} V_{0,0,0}E(H) \equiv E(H)(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}) + \hat{V}_{0,0,0}E(H) = \varphi_{0,0,0}, \\ (V_{1,0,0}E(H))(x) \equiv E(H)_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1}) + (\hat{V}_{1,0,0}E(H))(x) = \varphi_{1,0,0}(x), x\in G_1, \\ (V_{0,1,0}E(H))(y) \equiv E(H)_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1}) + (\hat{V}_{0,1,0}E(H))(y) = \varphi_{0,1,0}(y), y\in G_2, \\ (V_{0,0,1}E(H))(z) \equiv E(H)_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z) + (\hat{V}_{0,0,1}E(H))(z) = \varphi_{0,0,1}(z), z\in G_3, \\ (V_{1,1,0}E(H))(x,y) \equiv E(H)_{xy}(x,y,\sqrt{z_0z_1}) + (\hat{V}_{1,1,0}E(H))(x,y) = \varphi_{1,1,0}(x,y), (x,y)\in G_1\times G_2, \\ (V_{0,1,1}E(H))(y,z) \equiv E(H)_{yz}(\sqrt{x_0x_1},y,z) + (\hat{V}_{0,1,1}E(H))(y,z) = \varphi_{0,1,1}(y,z), (y,z)\in G_2\times G_3, \\ (V_{1,0,1}E(H))(x,z) \equiv E(H)_{xz}(x,\sqrt{y_0y_1},z) + (\hat{V}_{1,0,1}E(H))(x,z) = \varphi_{1,0,1}(x,z), (x,z)\in G_1\times G_3, \end{cases}$$

for the operators rather small in the norm
$\hat{V} = (\hat{V}_{0,0,0},\hat{V}_{1,0,0},\hat{V}_{0,1,0},\hat{V}_{0,0,1},\hat{V}_{1,1,0},\hat{V}_{0,1,1},\hat{V}_{1,0,1},\hat{V}_{1,1,1}): W_p^{(1,1,1)}(G)\to E_p^{(1,1,1)}$
everywhere is well-defined solvable. We will study the problem (8) by means of the integral representation (4) functions

$$E(H)(x,y,z)\in W_p^{(1,1,1)}(G).$$

Theorem. Let the operator $\hat{V}: W_p^{(1,1,1)}(G)\to E_p^{(1,1,1)}$, $1\le p\le\infty$ be rather small in the norm, and in the case $p=\infty$ has a conjugate acting in $E_1^{(1,1,1)}$. Then the problem (8) is ewerywhere well-defined solvable.

Proof. Since the operator (8) is an homeomorphism from $E_p^{(1,1,1)}$ to $W_p^{(1,1,1)}(G)$, then each function $E(H)(x,y,z)\in W_p^{(1,1,1)}(G)$ can be identified by the element

$$\begin{gathered} b = E(H)(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}), \\ E(H)_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1}), \\ E(H)_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1}), \\ E(H)_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z), \\ E(H)_{xy}(x,y,\sqrt{z_0z_1}), \\ E(H)_{yz}(\sqrt{x_0x_1},y,z), \\ E(H)_{xz}(x,\sqrt{y_0y_1},z), \\ E(H)_{xyz}(x,y,z) \end{gathered} \tag{9}$$

from $E_p^{(1,1,1)}$. Therefore, the problem (8) is equivalent to the canonical operator eqation

$$\boldsymbol{VQb \equiv b + \hat{V}Qb = \varphi}, \tag{10}$$

where
$\varphi = (\varphi_{0,0,0},\varphi_{1,0,0},\varphi_{0,1,0},\varphi_{0,0,1},\varphi_{1,1,0},\varphi_{0,1,1},\varphi_{1,0,1},\varphi_{1,1,1})\in E_p^{(1,1,1)}$,

$V=\left(V_{0,0,0},V_{1,0,0},V_{0,1,0},V_{0,0,1},V_{1,1,0},V_{0,1,1},V_{1,0,1},V_{1,1,1}\right).$

In what follows, let $f=\left(f_{0,0,0},f_{1,0,0},f_{0,1,0},f_{0,0,1},f_{1,1,0},f_{0,1,1},f_{1,0,1},f_{1,1,1}\right)\in E_q^{(1,1,1)}$ be some linear bounded functional in $E_p^{(1,1,1)}$, where 1/p + 1/q = 1. Since $\hat{V}$ has the conjugate $\hat{V}^*=\left(\hat{\omega}_{0,0,0},\hat{\omega}_{1,0,0},\hat{\omega}_{0,1,0},\hat{\omega}_{0,0,1},\hat{\omega}_{1,1,0},\hat{\omega}_{0,1,1},\hat{\omega}_{1,0,1},\hat{\omega}_{1,1,1}\right)$ acting in $E_q^{(1,1,1)}$, then for all $E(H)(x,y,z)\in W_p^{(1,1,1)}(G)$ the following identity is valid

$$f\left(\hat{V}u\right)\equiv\left(\hat{V}_{0,0,0}u\right)f_{0,0,0}+\int_{\sqrt{x_0x_1}}^{x_1}\left(\hat{V}_{1,0,0}u\right)f_{1,0,0}(x)(x)dx+ \tag{11}$$
$$+\int_{\sqrt{y_0y_1}}^{y_1}\left(\hat{V}_{0,1,0}u\right)f_{0,1,0}(y)dy+\int_{\sqrt{z_0z_1}}^{z_1}\left(\hat{V}_{0,0,1}u\right)(z)f_{0,0,1}(z)dz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\left(\hat{V}_{1,1,0}E(H)\right)(x,y)f_{1,1,0}(x,y)dxdy+$$
$$+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}\left(\hat{V}_{0,1,1}E(H)\right)f_{0,1,1}(y,z)dydz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1}\left(\hat{V}_{1,0,1}E(H)\right)(x,z)f_{1,0,1}(x,z)dxdz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}\left(\hat{V}_{1,1,1}E(H)\right)(x,y,z)f_{1,1,1}(x,y,z)dxdydz=$$
$$=E(H)(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})\left(\hat{\omega}_{0,0,0}f\right)+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}E(H)_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1})\left(\hat{\omega}_{1,0,0}f\right)(x)dx+$$
$$+\int_{\sqrt{y_0y_1}}^{y_1}E(H)_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1})\left(\hat{\omega}_{0,1,0}f\right)(y)dy+$$
$$+\int_{\sqrt{z_0z_1}}^{z_1}E(H)_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z)\left(\hat{\omega}_{0,0,1}f\right)(z)dz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}E(H)_{xy}(x,y,\sqrt{z_0z_1})(\hat{\omega}_{1,1,0}f)(x,y)dxdy+$$
$$+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}E(H)_{yz}(\sqrt{x_0x_1},y,z)(\hat{\omega}_{0,1,1}f)(y,z)dydz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1}E(H)_{xz}(x,\sqrt{y_0y_1},z)(\hat{\omega}_{1,0,1}f)(x,z)dxdz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}E(H)_{xyz}(x,y,z)(\hat{\omega}_{1,1,1}f)(x,y,z)dxdydz.$$

It follows from the identity (11) that for all $f\in E_q^{(1,1,1)}$ and $E(H)\in W_p^{(1,1,1)}(G)$ we have

$$f(VE(H))=E(H)(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})\left(f_{0,0,0}+\hat{\omega}_{0,0,0}f\right)+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}E(H)_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1})\left(f_{1,0,0}(x)+\hat{\omega}_{1,0,0}f\right)(x)dx+$$
$$+\int_{\sqrt{y_0y_1}}^{y_1}E(H)_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1})\left(f_{0,1,0}(y)+\hat{\omega}_{0,1,0}f\right)(y)dy+$$
$$+\int_{\sqrt{z_0z_1}}^{z_1}E(H)_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z)\left(f_{0,0,1}(z)+\hat{\omega}_{0,0,1}f\right)(z)dz+ \tag{12}$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}E(H)_{xy}(x,y,\sqrt{z_0z_1})(f_{1,1,0}(x,y)+\hat{\omega}_{1,1,0}f)(x,y)dxdy+$$
$$+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}E(H)_{yz}(\sqrt{x_0x_1},y,z)(f_{0,1,1}(y,z)+\hat{\omega}_{0,1,1}f)(y,z)dydz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1}E(H)_{xz}(x,\sqrt{y_0y_1},z)(f_{1,0,1}(x,z)+\hat{\omega}_{1,0,1}f)(x,z)dxdz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}E(H)_{xyz}(x,y,z)(f_{1,1,1}(x,y,z)+\hat{\omega}_{1,1,1}f)(x,y,z)dxdydz$$

It follows from the equality (12) that the operator V has a conjugate V^* that acts in the space $E_q^{(1,1,1)}$. Moreover,

$$V^*f=\left(f_{0,0,0}+\hat{\omega}_{0,0,0}f,\ f_{1,0,0}(x)+\left(\hat{\omega}_{1,0,0}f\right)(x),\ f_{0,1,0}(y)+\left(\hat{\omega}_{0,1,0}f\right)(y),\right.$$
$$f_{0,0,1}(z)+\left(\hat{\omega}_{0,0,1}f\right)(z),\ f_{1,1,0}(x,y)+\left(\hat{\omega}_{1,1,0}f\right)(x,y),\ f_{0,1,1}(y,z)+\left(\hat{\omega}_{0,1,1}f\right)(y,z),$$
$$\left.f_{1,0,1}(x,z)+\left(\hat{\omega}_{1,0,1}f\right)(x,z),\ f_{1,1,1}(x,y,z)+\left(\hat{\omega}_{1,1,1}f\right)(x,y,z)\right)=f+\hat{V}^*f.$$

Therefore, we can write the equation $V^*f=\psi$ in the form

$$V^*f=f+\hat{V}^*f=\psi, \tag{13}$$

where

$$\psi=\left(\psi_{0,0,0},\psi_{1,0,0},\psi_{0,1,0},\psi_{0,0,1},\psi_{1,1,0},\psi_{0,1,1},\psi_{1,0,1},\psi_{1,1,1}\right)\in E_q^{(1,1,1)}.$$

Using the identity (11), we can show that in the case $1\le p<\infty$ the operator $\hat{V}^*$ is a conjugate for $\hat{V}Q$, and in the case $1<p\le\infty$ the operator $\hat{V}Q$ is a conjugate for $\hat{V}^*$. Therefore, in all the cases $\left\|\hat{V}Q\right\|=\left\|\hat{V}^*\right\|$.

Obviously, if $f\in E_q^{(1,1,1)}$ is the solution of the equation (13), then

$$\|f\|_{E_q^{(1,1,1)}}\le\left\|\hat{V}^*\right\|\|f\|_{E_q^{(1,1,1)}}+\left\|\hat{V}^*f\right\|_{E_q^{(1,1,1)}}. \tag{14}$$

Choosing $\Delta=\left\|\hat{V}^*\right\|<1$ from (14) we obtain

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

$$\|f\|_{E_q^{(1,1,1)}} \le M^* \left\|\hat{V}^* f\right\|_{E_q^{(1,1,1)}},\ M^* = const.$$

Under the same condition, using (10), we can show that

$$\|E(H)\|_{W_p^{(1,1,1)}} \le M\|VE(H)\|_{E_p^{(1,1,1)}}, M = const \quad (15)$$

Inequality (15) shows that the operator V of the problem (8) is a homeomorphism from $W_p^{(1,1,1)}(G)$ to $E_p^{(1,1,1)}(G)$, i.e. the problem (8) is ewery where well-defined solvable. The theorem is proved.

There is a great arbitrariness in the choice of the operator $\hat{V}$ in problem (8). Therefore, problem (8) can be used as a source for obtaining new classes of well-posed boundary value problems for 3*D* (three-dimensional) Bianchi type equations.

IV. CALCULATION OF THE ELECTRIC FIELD OF AN MICROWAVE RECTANGULAR WAVEGUIDE

Based on the finite difference method, calculations were performed and the electromagnetic field strengths of a microwave rectangular waveguide with a nonlinear medium operating in *E*-type and *H*-type waves in the frequency range of 4,9-7,05 *GHz* were determined. The obtained numerical values are given in Table 1 and Table 2. Figure 2-6 shows 3*D* models of the electric field strength distribution for an *E*-type wave of a rectangular waveguide operating in the frequency range of 4,9-7,05 *GHz*. Figure 7-11 shows 3*D* models of the magnetic field strength distribution for an *E*-type wave of a rectangular waveguide operating in the frequency range of 4,9-7,05 *GHz*.

For the E-type wave

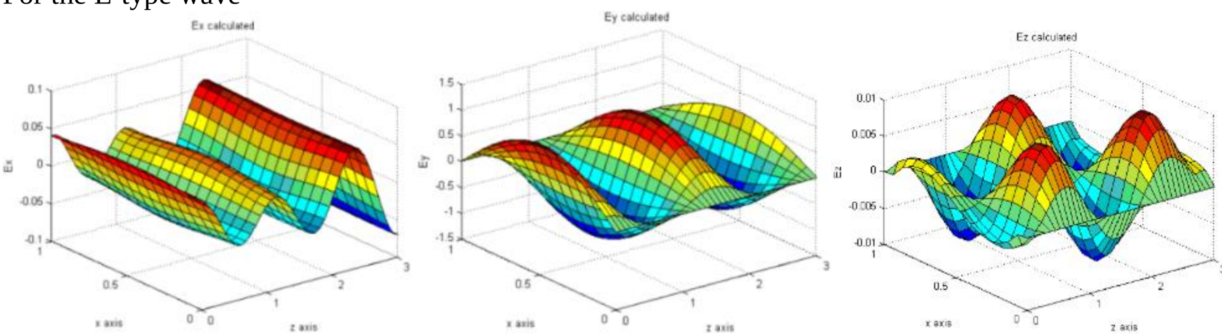

Fig. 2. Components of the field inside the waveguide in case $\chi = 0{,}01$.

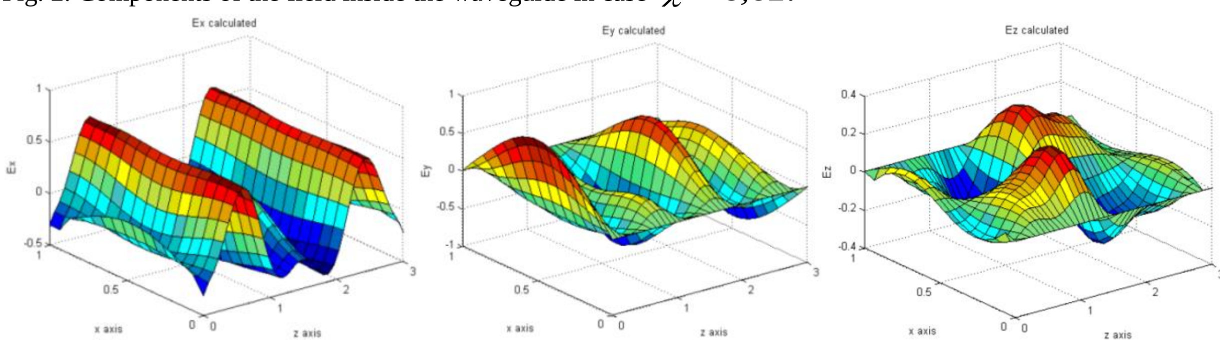

Fig. 3. Components of the field inside the waveguide in case $\chi = 0{,}5$.

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

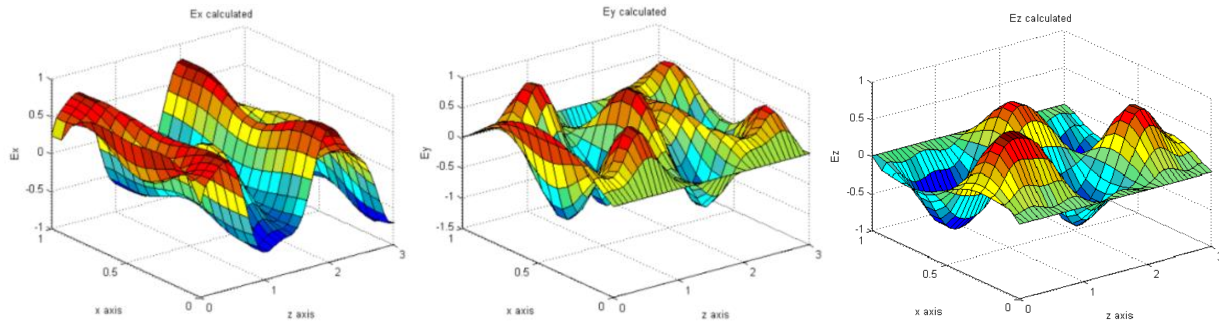

Fig. 4. Components of the field inside the waveguide in case $\chi = 0{,}8$.

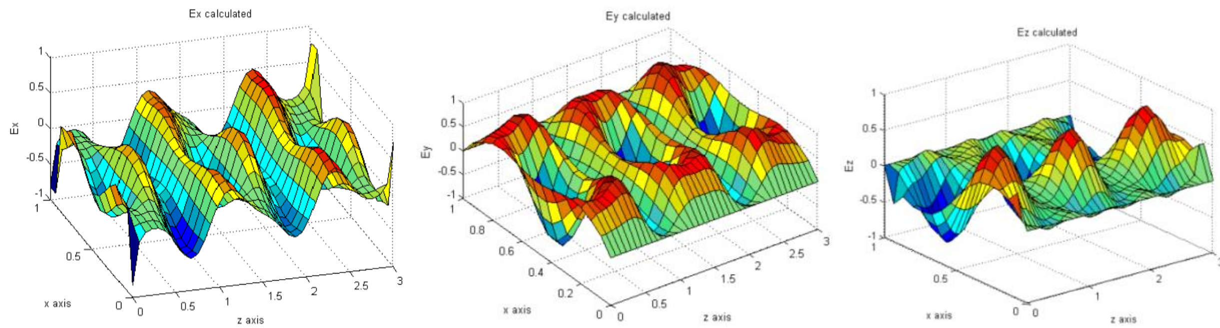

Fig. 5. Components of the field inside the waveguide in case $\chi = 1{,}01$.

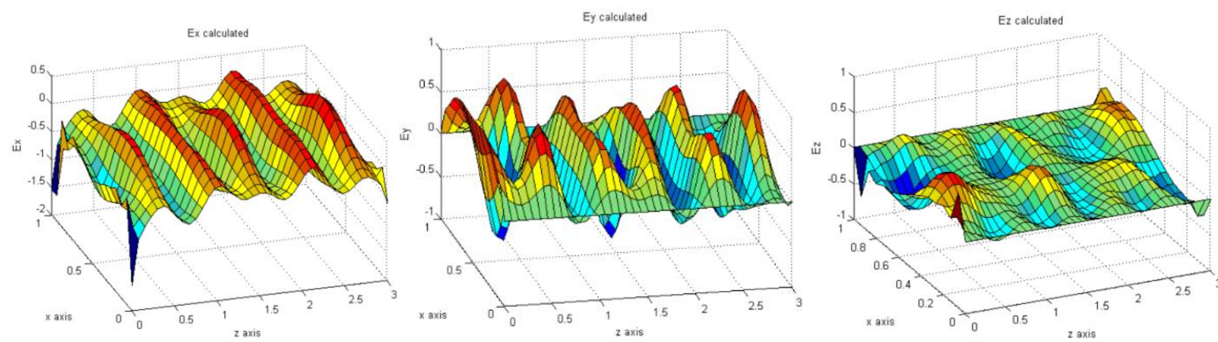

Fig. 6. Components of the field inside the waveguide in case $\chi = 1{,}1$.

For the *H*-type wave

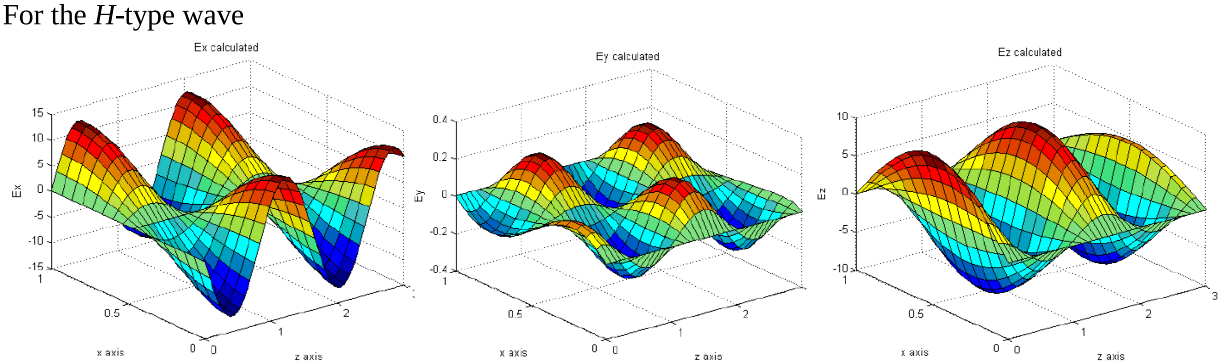

Fig. 7. Components of the field inside the waveguide in case $\chi = 0{,}01$.

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

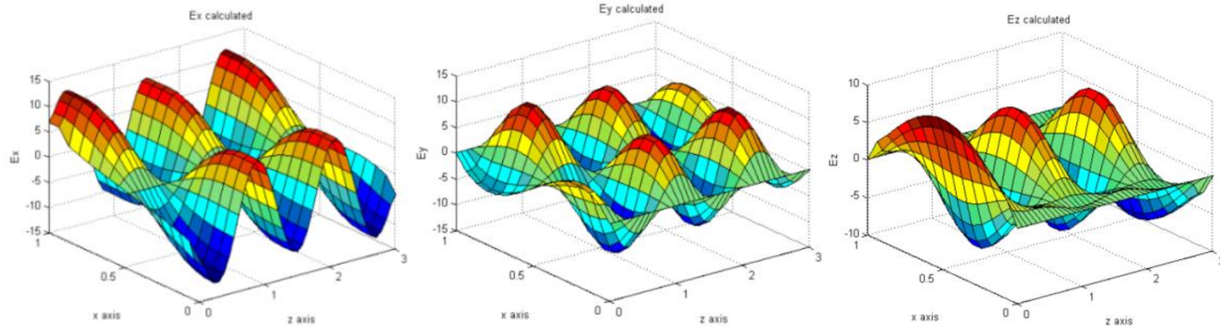

Fig. 8. Components of the field inside the waveguide in case $\chi = 0{,}5$.

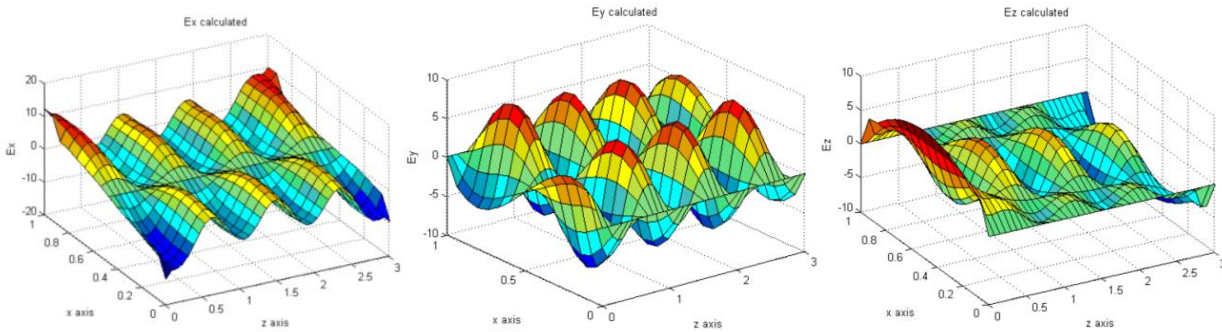

Fig. 9. Components of the field inside the waveguide in case $\chi = 0{,}8$.

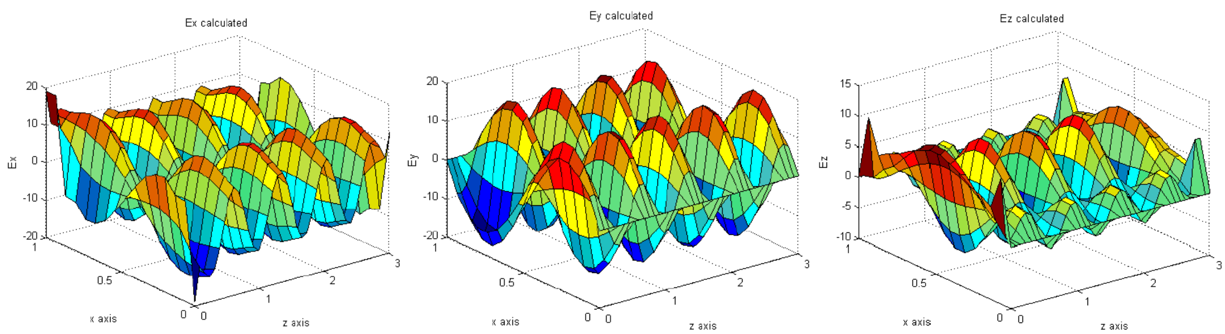

Fig. 10. Components of the field inside the waveguide in case $\chi = 1{,}01$.

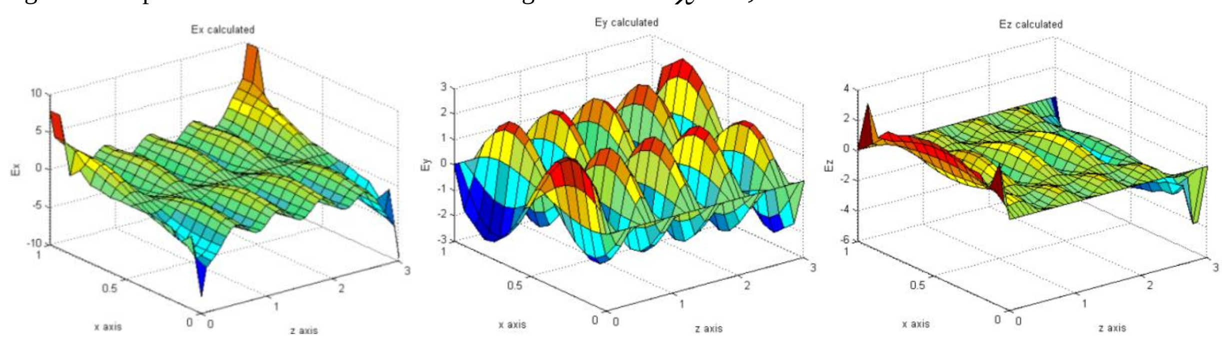

Fig. 11. Components of the field inside the waveguide in case $\chi = 1{,}1$.

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

V. DEVELOPMENT OF A MEASURING DEVICE FOR THE EXPERIMENTAL STUDY OF THE ELECTROMAGNETIC FIELD OF MICROWAVE WAVEGUIDE

The structural scheme and general appearance of the proposed measuring device for the investigation of the electromagnetic field in the waveguide are shown in Figure 12 and Figure 13, respectively. All nodes included in the waveguide tract of the measuring device are made on the basis of microwave rectangular waveguide with a cross-sectional area of 40x20 *mm*. The electromagnetic wave is supplied to the line through an microwave with a Hann diode. This generator is fed through a power supply unit. A ferrite valve is placed at the input part of the measuring device to agree the input with the load. After the attenuator, which is used to adjust the power level of the wave, the measuring line of the waveguide is placed. With the help of the measurement line, standing wave coefficient and distribution of the electromagnetic field in the load, as well as the values of the electric and magnetic field intensity vector of the waveguide are determined.

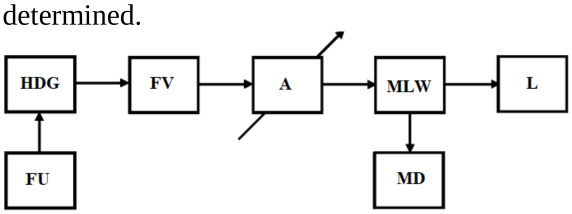

Fig. 12. Structural diagram of the measuring setup for experimental study of the electromagnetic field in a microwave waveguide: HDG – Hann diode generator; FV – ferrite valve; A – attenuator; MLW – measuring line of waveguide; L – load; FU – food unit; MD – measuring device.

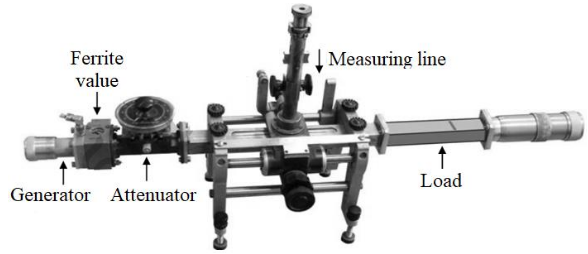

Fig. 13. General view of the measuring setup for experimental study of the electromagnetic field in a microwave waveguide.

The measuring device connected to the output of the measuring line detector is a constant voltage microvoltmeter. Table 1 shows a comparison of the results of existing works [1-5] and those obtained in this work. The comparisons were made using the following parameters of the microwave rectangular waveguide: limit power, releasable power, extinction coefficient, standing wave coefficient, refractive coefficient, special quality, the phase of the reflection coefficient, characteristic resistance. As can be seen from Table 1, the results obtained in this work are in good agreement with existing works [1-5]. The relative error between theoretical and experimental results for the electric field of the microwave rectangular waveguide is shown in Table 2, the relative error between theoretical and experimental results for the magnetic field of the microwave rectangular waveguide is shown in Table 3. Comparative analysis and error assessment of the results given in these tables show that the relative errors between theoretical and experimental results do not exceed 4%. This is also considered satisfactory for the microwave range.

Table 1: Comparison of the results of existing works and this work on the parameters of a microwave rectangular waveguide.

References	Limit power, *Wt*	Releasable power, *Wt*	Extinction coefficient, *dB/m*	Standing wave coefficient	Refractive coefficient	Special quality	The phase of the reflection coefficient, *rad/m*	Characteristic resistance, *Ohm*
[1]	4020,1	1209	0,0209	1,026	0,0091	4496,6	39,2π	377,7
[2]	4017,6	1218	0,0208	1,028	0,0093	4497,3	39,1π	377,9
[3]	4005,5	1210	0,0203	1,022	0,0096	4499,5	40,2π	376,1
[4]	4018,3	1207	0,0207	1,023	0,0097	4493,7	40,3π	378,3
[5]	4015,7	1205	0,0202	1,019	0,0090	4492,3	40,6π	378,1
This work	4010,5	1211	0,0201	1,021	0,0098	4497,2	39,9π	376,7

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

Table 2: Relative error between theoretical and experimental results for the electric field of an microwave rectangular waveguide.

Number of elementary regions	Experimental values of electric field intensity in elementary regions, (10^5), *V/m*				The values obtained from the calculation of the electric field intensity in elementary regions by the finite difference method, (10^5), *V/m*				Relative error between theoretical and experimental results $\left\|X_{exp.}-X_{cal.}\right\|/\left\|X_{exp.}\right\|\cdot 100\%$			
	E-wave		*H*-wave		*E*-wave		*H*-wave		*E*-wave		*H*-wave	
	Ex	*Ey*	*Ex*	*Ey*	*Ex*	*Ey*	*Ex*	*Ey*	*Ex*	*Ey*	*Ex*	*Ey*
1	2,25	4,51	4,52	2,55	2,18	4,45	4,45	2,45	3,11	1,33	1,54	3,92
2	3,52	5,18	1,59	2,61	3,48	5,24	1,6	2,59	1,13	1,15	0,62	0,76
3	4,10	5,30	1,38	3,42	4,07	5,28	1,43	3,39	0,73	0,37	3,62	0,87
4	4,50	5,29	5,50	4,22	4,47	5,29	5,42	4,13	0,66	0	1,45	2,13
5	4,41	5,41	5,49	3,36	4,44	5,36	5,51	3,44	0,68	0,92	0,36	2,38
6	4,42	6,11	4,81	3,44	4,43	6,07	4,78	3,37	0,22	0,65	0,62	2,03
7	4,49	6,51	2,09	2,39	4,43	6,42	2,11	2,42	1,33	1,38	0,95	1,25
8	2,91	4,66	1,40	1,42	2,83	4,75	1,37	1,40	2,74	1,93	2,14	1,40
9	3,78	4,19	4,19	1,21	3,85	4,27	4,21	1,19	1,85	1,90	0,47	1,65
10	4,22	5,51	5,29	1,26	4,15	5,47	5,21	1,23	1,65	0,72	1,51	2,38
11	4,13	6,11	6,61	1,62	4,19	6,03	6,52	1,60	1,45	1,30	1,36	1,23
12	3,17	6,30	4,22	1,57	3,16	6,25	4,12	1,54	0,31	0,79	2,36	1,91
13	2,69	6,18	4,21	1,81	2,67	6,26	4,11	1,79	0,74	1,29	2,37	1,10
14	2,62	6,32	7,26	2,25	2,55	6,29	7,31	2,19	2,67	0,47	0,68	2,66
15	2,50	5,63	7,62	3,71	2,43	5,59	7,52	3,61	2,8	0,71	1,31	2,69
16	2,35	5,09	1,53	1,59	2,27	5,14	1,49	1,57	3,4	0,98	2,61	1,25
17	2,88	5,19	1,44	1,21	2,91	5,12	1,42	1,19	1,04	1,37	1,38	1,65
18	3,23	1,21	2,48	4,20	3,16	1,20	2,51	4,1	2,16	0,82	1,20	2,35
19	4,50	4,42	3,21	5,52	4,37	4,37	3,19	5,49	2,88	1,13	0,62	0,54
20	5,31	4,81	4,51	6,51	5,24	4,78	4,42	6,49	1,31	0,62	1,99	0,30

Table 3: Relative error between theoretical and experimental results for the magnetic field of an microwave rectangular waveguide.

Number of elementary regions	Experimental values of magnetic field intensity in elementary regions, *A/m*				The values obtained from the calculation of the magnetic field intensity in elementary regions by the finite difference method, *A/m*				Relative error between theoretical and experimental results $\left\|X_{exp.}-X_{cal.}\right\|/\left\|X_{exp.}\right\|\cdot 100\%$			
	E-wave		*H*-wave		*E*-wave		*H*-wave		*E*-wave		*H*-wave	
	Hx	*Hy*	*Hx*	*Hy*	*Hx*	*Hy*	*Hx*	*Hy*	*Hx*	*Hy*	*Hx*	*Hy*
1	51,82	24,31	54,02	28,30	51,86	24,2	53,6	27,94	0,08	0,45	0,78	1,27
2	67,84	43,24	25,43	35,63	67,11	44,25	24,5	36,3	1,08	2,34	3,66	1,88
3	44,53	25,63	35,82	77,12	43,56	25,43	36,57	76,23	2,18	0,78	2,09	1,15
4	63,72	35,42	46,22	45,74	64,16	34,98	44,89	46,98	0,69	1,22	2,88	2,71
5	11,64	85,81	66,14	62,71	11,34	86,78	65,79	63,39	2,58	1,13	0,53	1,08
6	52,61	43,70	34,53	64,74	53,12	44,9	34,79	65,15	0,97	2,75	0,75	0,63
7	54,74	38,92	76,32	22,62	55,56	39,19	74,76	22,37	1,50	0,69	2,04	1,11
8	61,23	32,13	24,31	34,73	60,47	33,18	24,26	33,71	1,34	3,27	0,21	2,94
9	63,41	25,24	12,11	37,72	62,12	25,17	11,98	38,18	2,03	0,28	1,07	1,22
10	53,20	44,34	32,63	11,73	51,45	43,45	33.76	11,56	3,29	2,01	3,46	1,45
11	52,92	45,93	36,22	33,71	51,9	44,86	35,69	32,96	1,93	2,33	1,46	2,22
12	43,54	47,11	74,72	35,44	42,45	46,58	73,72	34,48	2,39	1,13	1,34	2,71
13	34,73	52,53	36,14	20,60	35,19	53,56	35,43	20,11	1,32	1,96	1,96	2,38
14	55,72	66,90	66,23	41,91	56,61	67,5	67,31	40,81	1,6	0,9	1,63	2,62
15	47,52	31,72	34,33	42,53	48,13	31,34	34,53	43,12	1,28	1,2	0,58	1,39
16	59,20	43,71	34,10	29,52	60,16	42.95	33,99	30,15	1,62	1,74	0,32	2,13
17	79,34	62,94	64,32	10,23	80,78	61,89	64,44	10,35	1,81	1,67	0,19	1,17
18	61,71	44,43	75,83	37,71	62,34	43,56	76,45	38,48	1,02	1,96	0,82	2,04
19	61,82	44,21	41,94	29,54	60,45	43,56	41,25	30,05	2,22	1,47	1,65	1,73
20	49,51	25,83	41,11	21,63	50,23	26,21	40,38	22,38	1,45	1,47	1,78	3,47

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

VI. CONCLUSION

New mathematical models have been developed in the Cartesian coordinate system of the electromagnetic field of a rectangular waveguide operating in the frequency range of 4,9-7,05 *GHz*, taking into account the nonlinearity of the medium and types of waves, and effective algorithms for solving the models have been proposed, which has improved the electrical, magnetic, structural and operational parameters and characteristics of the microwave rectangular waveguide. For *E*-type and *H*-type waves, 3*D* models of the distribution of electromagnetic field strengths in elementary regions of a rectangular waveguide operating in the frequency range of 4,9-7,05 *GHz* have been developed. Experimental devices and functional circuits for measuring the parameters of a microwave path, including electric and magnetic fields, a rectangular waveguide in a nonlinear state of the medium are proposed, and the waveguide parameters are experimentally determined. Comparison of theoretical and experimental results of electric and magnetic field strengths showed that the relative error for these parameters is 4%. This proved the adequacy of the theoretical and experimental results obtained.

ACKNOWLEDGMENT

The authors would like to thank the editor and anonymous reviewers for constructive, valuable suggestions and comments on the work.

REFERENCES

[1] S. Zhang, B. Cheng, Z. Jia, Z. Zhao, X. Jin, Z. Zhao, and G. Wu, "The art of framework construction: hollow-structured materials toward high-efficiency electromagnetic wave absorption," *Advanced Composites and Hybrid Materials*, vol. 5, pp. 1658-1698, November 2022.

[2] A. Bhardwaj, D. Pratap, K.V. Srivastava, and A. Ramakrishna, "Analytical analysis of inhomogeneous and anisotropic metamaterial cylindrical waveguides using transformation matrix method," *Journal of Electromagnetic Waves and Applications*, vol. 37(1), pp. 53-68, December 2023.

[3] Q.-W. Lin, S. Alkaraki, H. Wong, and J.R. Kelly, "A Wideband Circularly Polarized Antenna Based on Anisotropic Metamaterial," *IEEE Transactions on Antennas and Propagation*, vol. 71(2), pp. 1254-1262, December 2022.

[4] P. Karimi, B. Rejaei, and A. Khavasi, "Spin-polarized unidirectional cylindrical waveguide in bianisotropic media," *Optics Express*, vol. 28(16), pp. 24022-24036, December 2020.

[5] C.J.M. Barker, N.D. Zanche, and A.K. Iyer, "Dispersion and Polarization Control in Below-Cutoff Circular Waveguides Using Anisotropic Metasurface Liners," *IEEE Transactions on Microwave Theory and Techniques*, vol. 71(8), pp. 3392-3403, April 2023.

[6] G.P. Zouros, and K. Katsinos, "EM Scattering by Gyrotropic Circular Cylinders With Arbitrarily Oriented External Magnetic Bias," *IEEE Transactions on Antennas and Propagation*, vol. 71(1), pp. 1174-1179, November 2022.

[7] E. Smolkin, and M. Snegur, "Mathematical Theory of Leaky Waves in an Anisotropic Waveguide," *Lobachevskii Journal of Mathematics*, vol. 43, pp. 1285-1292, August 2022.

[8] J. Liu, C.F. Tong, W. Jiang, and Q.H. Liu, "3-D NMM Method for Fully Anisotropic and Nonreciprocal Media," *IEEE Transactions on Microwave Theory and Techniques*, vol. 70(7), pp. 3428-3441, May 2022.

[9] X.F. Hu, Y.R. Fan, S.G. Deng, X.Y. Yuan, and H.T. Li, "Electromagnetic logging response in multilayered formation with arbitrary uniaxially electrical anisotropy," *IEEE Trans. Geosci. Remote Sens*., vol. 58(3), pp. 1134-1146, December 2019.

[10] V. Kamra, and A. Dreher, "Analysis of circular and noncircular waveguides and striplines with multilayered uniaxial anisotropic medium," *IEEE Trans. Microw. Theory Techn*., vol. 67(2), pp. 548-591, December 2018.

[11] Y. Xu, et al., "Design and Test of Broadband Rectangular Waveguide TE_{10} to Circular Waveguide TE_{21} and TE_{01} Mode Converters," *IEEE Transactions on Electron Devices*, vol. 66(8), pp. 3573-3579, June 2019.

[12] S.N. Galyamin, and V.V. Vorobev, "Diffraction at the Open End of Dielectric-Lined Circular Waveguide," *IEEE Transactions on Microwave Theory and Techniques*, vol. 70(6), pp. 3087-3095, April 2022.

[13] Z. Wu, M. Wang, X. Liao, Y. Pu, L. Wang, and W. Jiang, "Oversized Multimode Waveguide Filter for Circular TE_{01} Mode High Power Transmission Lines," *IEEE Transactions on Microwave Theory and Techniques*, vol. 71(8), pp. 3552-3560, February 2023.

[14] B. Michal, L. Balewski, A. Lamecki, M. Mrozowski, and J. Galdeano, "A Circular Waveguide Dual-Mode Filter With Improved Out-of-Band Performance for Satellite Communication

INTERNATIONAL JOURNAL OF MICROWAVE AND OPTICAL TECHNOLOGY,
VOL.19, NO.6, NOVEMBER 2024

Systems," *IEEE Microwave and Wireless Components Letters*, vol. 32(12), pp. 1403-1406, August 2022.

[15] F. Kamrath, E. Polat, S. Matic, C. Schuster, D. Miek, and H. Tesmer, "Bandwidth and Center Frequency Reconfigurable Waveguide Filter Based on Liquid Crystal Technology," *IEEE Journal of Microwaves*, vol. 2(1), pp. 134-144, October 2021.

[16] Y. Liu, K.-D. Xu, J. Li, Y.-J. Guo, A. Zhang, and Q. Chen, "Millimeter-Wave E-Plane Waveguide Bandpass Filters Based on Spoof Surface Plasmon Polaritons," *IEEE Transactions on Microwave Theory and Techniques*, vol. 70(10), pp. 4399-4409, August 2022.

[17] T.M. Nguyen, T.K.T. Nguyen, D.T. Phan, D.T. Le, D.L. Vu, and T. Quyn, "Ultra-Wideband and Lightweight Electromagnetic Polarization Converter Based on Multiresonant Metasurface," *IEEE Access*, vol. 10, pp. 92097-92104, August 2022.

[18] A.N. Bogolyubov, A.I. Erokhin, and M.I. Svetkin, "Mathematical Modeling of Impedance Waveguide Systems," *Moscow University Physics Bulletin*, vol. 74, pp. 227-232, August 2019.

[19] H. Zhou, X.-H. Wang, L. Xiao, and B.-Z. Wang, "Efficient EDM-PO Method for the Scattering From Electrically Large Objects With the High-Order Impedance Boundary Condition," *IEEE Transactions on Antennas and Propagation*, vol. 70(9), pp. 8242-8249, May 2022.

[20] B. Stupfel, P. Payen, and O. Lafitte, "A well-posed and effective high-order Impedance Boundary Condition for the time-harmonic scattering problem from a multilayer coated 3-D object," *Progress In Electromagnetics Research B*, Vol. 94, pp. 127-144, May 2021.

[21] B. Stupfel, "Impedance boundary conditions for finite planar and curved frequency selective surfaces embedded in dielectric layers," *IEEE Trans. Antennas Propagat.*, vol. 53(11), pp. 3654-3663, November 2005.

[22] J.H. Beggs, R.J. Luebbers, K.S. Yee, and K.S. Kunz, "Finite-difference time-domain implementation of surface impedance boundary conditions," *IEEE Transactions on Antennas and Propagation*, vol. 40(1), pp. 49-56, August 1992.

[23] C. Tao, Y. Zhong, and H. Liu, "Quasinormal Mode Expansion Theory for Mesoscale Plasmonic Nanoresonators: An Analytical Treatment of Nonclassical Electromagnetic Boundary Condition," *Phys. Rev. Lett.*, vol. 129, 197401, November 2022.

[24] S. Shams, F. Mohajeri, and M. Movahhedi, "A Dispersive Formulation of Time-Domain Meshless Method With PML Absorbing Boundary Condition for Analysis of Left-Handed Materials," *IEEE Transactions on Antennas and Propagation*, vol. 70(8), pp. 7328-7333, April 2022.

[25] H. Bao, T. Zhang, D. Ding, R. Chen, and D.H. Werne, "Generalized Periodic Boundary Conditions for DGTD Analysis of Arbitrary Skewed Periodic Structures," *IEEE Transactions on Microwave Theory and Techniques*, vol. 70(6), pp. 2989-2998, May 2022.

[26] Y. Lai, and Z. Wang, "Unique interface reflection phenomena tailored by nanoscale electromagnetic boundary conditions," *Optics Express*, vol. 30(18), pp. 33112-33123, August 2022.

[27] M.M.M. Ali, S.I. Shams, M. Elsaadany, K. Wu, "Stripline Y-Junction Circulators: Accurate Model and Electromagnetic Analysis Based on Gaussian Field Distribution Boundary Conditions," *IEEE Transactions on Microwave Theory and Techniques*, vol. 71(1), pp. 146-155, August 2022.

[28] F. Wei, G. Chen, and W. Wang, "Finite-time stabilization of memristor-based inertial neural networks with time-varying delays combined with interval matrix method," *Knowledge-Based Systems*, vol. 230, 107395, October 2021.

[29] C. Regimantas, P. Kristina, and S. Mifodijus, "On the numerical solution for nonlinear elliptic equations with variable weight coefficients in an integral boundary conditions," *Nonlinear analysis: modelling and control*, vol. 26(4), pp. 738-758, October 2021.

[30] D. Sayad, F. Benabdelaziz, C. Zebiri, S. Daoudi, and R.A. Abd-Alhameed, "Spectral Domain Analysis of Gyrotropic Anisotropy Chiral Effect on the Input Impedance of a Printed Dipole Antenna," *Progress In Electromagnetics Research M*, vol. 51, pp. 1-8, October 2016.

[31] S. Amanatiadis, T. Ohtani, T. Zygiridis, Y. Kanai, and N. Kan, "Mode Propagation Analysis of Magnetically Biased Graphene Microstrips via an Efficient Finite-Difference Scheme," *IEEE Transactions on Magnetics*, vol. 58(9), pp. 454-467, May 2022.

[32] X. Zeng, and H. Yang, "A Class of Explicit Divergence-Free Methods for Maxwell's Equations With Dirichlet Boundary Conditions," *IEEE Access*, vol. 10, pp. 126188-126198, November 2022.

[33] R. Qiannan, U.Z. Ashraf, and Y. Jian, "Dual-Circularly Polarized Array Antenna Based on Gap Waveguide Utilizing Double-Grooved Circular Waveguide Polarizer," *IEEE Transactions on Antennas and Propagation*, vol. 70(11), pp. 10436-10444, August 2022.

[34] S. Corteel, J. Haglund, O. Mandelshtam, S. Mason, and L. Williams, "Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials," *Selecta Mathematica*, vol. 28, pp. 123-138, January 2022.

[35] I. Islamov, A. Safarli, "Design, Modelling and Research of an Antenna System for Transmitting and Receiving Information in Satellite Systems," *Transport and Telecommunication journal*,

vol. 24 (3), pp. 297-308, June 2023.

[36] I. Islamov, M. Hasanov, B. Ibrahimov, A. Movsumov, "Computer simulation, visualization, and synthesis of a digital antenna array used for transmitting and receiving signals in industrial applications," *International Journal of Microwave and Wireless Technologies*, vol. 15(10), pp. 1768-1780, April 2023.

[37] I.J. Islamov, and E.G. Ismibayli, "Experimental Study of Characteristics of Microwave Devices Transition from Rectangular Waveguide to the Megaphone," *IFAC-PapersOnLine*, vol. 51(30), pp. 477-479, September 2018.

[38] I.J. Islamov, E.G. Ismibayli, M.H. Hasanov, Y.G. Gaziyev, S.R. Ahmadova, and R.Sh. Abdullayev, "Calculation of the Electromagnetic Field of a Rectangular Waveguide with Chiral Medium," *Progress in Electromagnetics Research*, vol. 84, pp. 97-114, September 2019.

[39] I.J. Islamov, E.Z. Hunbataliyev, and A.E. Zulfugarli, "Numerical Simulation of Characteristics of Propagation of Symmetric Waves in Microwave Circular Shielded Waveguide with a Radially Inhomogeneous Dielectric Filling," *International Journal of Microwave and Wireless Technologies*, vol. 14(6), pp. 761-767, July 2021.

[40] I. Islamov, and E. Humbataliyev, "General Approaches to Solving Problems of Analysis and Synthesis of Directional Properties of Antenna Arrays," *Advanced Electromagnetics*, vol. 11(4), pp. 22-33, Octomber 2022.

[41] I.J. Islamov, "Modeling the Attenuation of a Microwave Signal, Taking Into Account the Specific Features of the Terrain for Unmanned Aerial Vehicles," *Internatıonal Journal of Mıcrowave and Optıcal Technology*, vol. 18(2), pp. 131-139, March 2023.

[42] R. Caroca, I. Kondrashuk, N. Merino and F. Nadal, "Bianchi spaces and their three-dimensional isometries as S-expansions of two-dimensional isometries," *Journal of Physics A: Mathematical and Theoretical*, vol. 46(22), 46 225201, May 2013.

[43] A.Y. Kamenshchik, "The Bianchi Classification of the Three-Dimensional Lie Algebras and Homogeneous Cosmologies and the Mixmaster Universe," *Physical and Mathematical Aspects of General Relativity*, pp. 93-137, November 2019.

[44] S. Azami, M. Jafari, A. Haseeb, A.A.H. Ahmadini, "Cross Curvature Solitons of Lorentzian Three-Dimensional Lie Groups," *Axioms*, vol. 13(4), 211, March 2024.

[45] F. Ma, Y. Cao, H. Du, et al., "Three-dimensional chromatin reorganization regulates B cell development during ageing," *Nat Cell Biol*, vol. 26, pp. 991-1002, June 2024.

[46] I.G. Mamedov, A.J. Abdullaeva, "On the correct solvability of the boundary-value problem in non-classical treatment on the geometric center of the domain for a integro-differential equation of 3D Bianchi," *Natural and Technology Sciences*, vol. 18(3), pp. 4-13, November 2018.

[47] I.G. Mamedov, A.J. Abdullaeva, "One 3D in the Geometrical Middle Problem in the Non-Classical Treatment for one 3D Bianchi integro-differential Equation with Non-Smooth Coefficients," *Caspian Journal of Applied Mathematics, Ecology and Economics*, vol. 6(1), pp. 73-81, May 2018.

[48] I.G. Mamedov, A.J. Abdullaeva, "On an optimal control problem for 3D Bianchi integro-differential equations with nonsmooth coefficients under conditions in the geometric middle of the domain," Informatics and Control Problems, vol. 40(1), pp. 32-40, May 2020.