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Spectral analysis of the indefinite non-self-adjoint Sturm–Liouville operator

Rakib Efendiev, Yusif Gasimov *

Baku Engineering University, Mathematics Department, H.Aliyev, 120, Baku, AZ0101, Azerbaijan
Azerbaijan University, Jeyhun Hajibeyli street 71, Baku, AZ1007, Azerbaijan
Institute of Mathematics and Mechanics, Ministry of Science and Education, B. Vahavzade street 9, Baku, AZ1148, Azerbaijan

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ABSTRACT

The study investigates the inverse scattering problem for the Schrodinger operator with complex potentials, considering indefinite discontinuous coefficients on the axis. Using the integral representation of the Jost solutions on the real and imaginary axes, solved the direct scattering problem. An additional study of the operator's spectrum was conducted, scattering data was introduced, and the eigenfunction expansion was obtained. Integral equations derived play a crucial role in solving the inverse problem and finally prove the uniqueness theorem for the solution.

1. Introduction

The propagation of plane-wave in the layered medium is described by the operator L which is given by the differential expression

$$l(y) \equiv \frac{1}{\rho(x)}\left[-y'' + q(x)\,y\right] \tag{1}$$

in the Hilbert space $L_2(-\infty,\infty)$.

It is assumed that the function (potential) $q(x)$ is complex-valued and fulfils condition

$$\int_{-\infty}^{\infty}(1+|x|)\,|q(x)|\,dx < \infty, \tag{2}$$

and function $\rho(x)$ has the form

$$\rho(x) = \begin{cases} 1 & for \quad x \geq 0 \\ -1 & for \quad x < 0 \end{cases} \tag{3}$$

The subject of this paper is the theory of inverse scattering problems i.e. recovery of operators from their scattering data.

Various types of inverse problems involve deducing certain properties of the given issue from a complete or partial understanding of its solution.[1–10] The distinction is somewhat unclear; however, it indicates that the contrast between a 'direct problem' and an 'inverse problem' is somewhat arbitrary and is based on the historical evolution of that specific problem. At this time, there is no general inverse spectral theory. Even the simplest cases require considerable ingenuity for their resolution, and none of the inverse problems can be posed unless a class of coefficients is specified in advance.

The problem under consideration is not only interesting in its own right but also because it can be reduced to other inverse problems in physics. After separating variables, the kinetic equation reduces to Sturm–Liouville eigenvalue problems with an indefinite function.

The use of complex potentials in the past has allowed for the modelling of neutron absorption. The imaginary potential signifies emission or absorption. For instance, in the crystal model, the complex potential is interpreted as representing a 'prepared' material. This material is assumed to be capable of absorption or emission.

The problem of inverse scattering (1)–(3) when $q(x) = 0$ was first considered by Belishev in,[11] where the inverse problem of reconstructing $\rho(x)$ from the frequencies and energies of its normalised characteristic vibrations were studied.

The most complete results for self-adjoint cases (i.e. for real potentials) in the theory of the inverse scattering problem at $\rho(x) = 1$ were obtained for the Sturm–Liouville operators $-y'' + q(x)\,y$ in the classic papers[12,13] and for $\rho(x) \neq 1$ in.[14,15]

Because the scattering function of a non-self-adjoint operator is not unitary and, additionally, can have pole-type singularities (the case of an operator with spectral singularities), generally speaking, it is not summable. Therefore, it is of particular interest to consider the inverse scattering problem (1)–(3).

Since $q(x)$ is complex and $\rho(x)$ changes its sign, the equation $ly = \lambda^2 y$ becomes of mixed type and the inverse problems are more difficult to study and the classical methods are either inapplicable or require essential modifications.

Inverse scattering problems for the non-self-adjoint Sturm–Liouville operator L defined by the differential expression (1) without discontinuities (i.e. $\rho(x) = 1$) were studied by Lyantse[3] on semiaxes and by Blashak in[16] on whole axes where the specific problems were considered in detail.

* Corresponding author.
E-mail address: yusif.gasimov@au.edu.az (Y. Gasimov).

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Problem (1)–(3) for the complex-valued, periodic and almost periodic potentials was considered in the papers.[14,17–20]

Properties of scattering data for the case when $q(x)$ is real and $\rho(x)$ is of the form (3) on half axes and the presence of discontinuity conditions essentially complicates even the study of the direct problem when its spectral properties are to be determined. Many studies deal with inverse problems for the Sturm–Liouville operator with different discontinuity conditions.ve been studied by Z. F.Abd-El-Reheem in.[21]

K. R. Mamedov, A. A Nabiev has studied direct and inverse scattering problems on the real axis for the operator L with real potentials $q(x)$ and $\rho(x)$, which is a positive step function, in.[15]

V.A. Yurko has dealt with inverse spectral problems on the finite interval and semi-axis for Sturm–Liouville operators with complex piece-wise-constant weights.[4,5]

Inverse problems for the complex Dirac operator with the jump conditions within the interval have been studied by R. Zhang, C.- F. Yang, and N.P. Bondarenko in.[6] They proved that Weyl-type function or two spectra can uniquely determine the potential on the whole interval.

The presence of discontinuity conditions essentially complicates even the study of the direct problem when its spectral properties are to be determined.

The paper's organization is as follows.

In Section 2, using the integral representation (5-6) of the Jost solutions of Eq. (4) on the real and imaginary axes, we solve the problem of direct scattering. In Section 3, we studied the spectrum of the operator L and introduced the scattering data. Section 4 is devoted to the eigenfunction expansion. In Section 5, we derived integral equations that play an important role in solving the inverse problem. Finally, Section 6 proves the uniqueness theorem for the solution of the inverse problem.

2. Special solutions of the equation $l(y) = \lambda^2 y$

Consider the equation

$$-y'' + q(x)\,y = \lambda^2 \rho(x)\,y \tag{4}$$

where $q(x)$ and $\rho(x)$ defined by conditions (2) and (3).

It is known[12] that solutions of Eq. (4) exist, holomorphic, are unique and can be represented in the following form

$$f_1^{\pm}(x,\lambda) = e^{\pm i\lambda x} + \int_x^{\infty} K_1(x,t)e^{\pm i\lambda t}dt \quad for \ \pm \operatorname{Im}\lambda \geq 0, x \geq 0, \tag{5}$$

$$f_2^{\pm}(x,\lambda) = e^{\pm\lambda x} + \int_{-\infty}^{x} K_2(x,t)e^{\pm\lambda t}dt \quad for \ \pm \operatorname{Re}\lambda \geq 0, x < 0, \tag{6}$$

$$\begin{gathered} \int_0^\infty \int_x^\infty [|K_1(x.t)|]^2 dt dx < \infty, \\ \int_{-\infty}^0 \int_{-\infty}^x [|K_2(x.t)|]^2 dt dx < \infty. \end{gathered} \tag{7}$$

For differentiable $q(x)$ kernels $K_1(x,t)$ and $K_2(x,t)$ satisfy the equations

$$\frac{\partial^2}{\partial x^2}K_1(x,t) - q(x)K_1(x,t) = \frac{\partial^2}{\partial t^2}K_1(x,t) \quad for \ x \geq 0, \tag{8}$$

$$\frac{\partial^2}{\partial x^2}K_2(x,t) - q(x)K_2(x,t) = \frac{\partial^2}{\partial t^2}K_2(x,t) \quad for \ x < 0, \tag{9}$$

where

$$K_1(x,t) = \frac{1}{2}\int_x^\infty q(t)\,dt \quad for \quad x \geq 0, \tag{10}$$

$$K_2(x,t) = \frac{1}{2}\int_{-\infty}^x q(t)\,dt \quad for \quad x < 0. \tag{11}$$

It is easy to verify if the kernels satisfy conditions (7)–(10) and the conditions at infinity

$$\begin{gathered} \lim_{(x+t)\to\infty} K'_{1x}(x,t) = \lim_{(x+t)\to\infty} K'_{1t}(x,t) = \\ = \lim_{(x+t)\to-\infty} K'_{2x}(x,t) = \lim_{(x+t)\to-\infty} K'_{2t}(x,t) = 0 \end{gathered} \tag{12}$$

then the functions $f_1^{\pm}(x,\lambda)$, $f_2^{\pm}(x,\lambda)$ can be constructed to obtain solutions for Eq. (4) with the potential $q(x)$

$$q(x) = \begin{cases} -2\frac{d}{dx}K_1(x,x) & for \ \ x \geq 0 \\ -2\frac{d}{dx}K_2(x,x) & for \ \ x < 0 \end{cases} \tag{13}$$

Let $[f(x), g(x)]$ denote the Wronskian of the functions $f(x), g(x)$

$$[f(x), g(x)] = f'(x)\,g(x) - f(x)\,g'(x). \tag{14}$$

Since the Wronskian of any two solutions of (4) does not depend on x, it coincides with their values for $x \to +\infty$ or $x \to -\infty$. As a result, considering of the formulas

$$\begin{aligned} &f_1^{\pm}(x,\lambda) = e^{\pm i\lambda x}(1 + o(1)), \ x \to +\infty, \\ &f_1^{\pm'}(x,\lambda) = e^{\pm i\lambda x}(\pm i\lambda + o(1)), \ x \to +\infty \\ &f_2^{\pm}(x,\lambda) = e^{\pm\lambda x}(1 + o(1)), \ x \to -\infty, \\ &f_2^{\pm'}(x,\lambda) = e^{\pm\lambda x}(\pm\lambda + o(1)), \ x \to -\infty \end{aligned}$$

we find

$$[f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)] = 2i\lambda \neq 0 \ for \ \operatorname{Im}\lambda = 0, \lambda \neq 0, \tag{15}$$

$$[f_2^{+}(x,\lambda), f_2^{-}(x,\lambda)] = 2\lambda \neq 0 \ for \ \operatorname{Re}\lambda = 0, \lambda \neq 0. \tag{16}$$

Therefore, the functions $f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)$ and the functions $f_2^{+}(x,\lambda), f_2^{-}(x,\lambda)$ are linearly independent solutions of the Eq. (4) if $\lambda \neq 0$, $\operatorname{Im}\lambda = 0$, $\operatorname{Re}\lambda = 0$ accordingly. Consequently, in the case $\operatorname{Im}\lambda = 0$ solution of Eq. (4) can be represented as a linear combination of solutions $f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)$ and for the case $\operatorname{Re}\lambda = 0$ as a linear combination of the solutions $f_2^{+}(x,\lambda), f_2^{-}(x,\lambda)$.

We have

$$\begin{cases} f_2^{+}(x,\lambda) = A(\lambda)\,f_1^{+}(x,\lambda) + C(\lambda)\,f_1^{-}(x,\lambda) \\ f_2^{-}(x,\lambda) = B(\lambda)\,f_1^{+}(x,\lambda) + D(\lambda)\,f_1^{-}(x,\lambda) \\ f_1^{+}(x,\lambda) = iD(\lambda)\,f_2^{+}(x,\lambda) - iC(\lambda)\,f_2^{-}(x,\lambda) \\ f_1^{-}(x,\lambda) = -iB(\lambda)\,f_2^{+}(x,\lambda) + iA(\lambda)\,f_2^{-}(x,\lambda) \end{cases} \tag{17}$$

where

$$\begin{cases} A(\lambda) = \frac{[f_2^{+}(x,\lambda), f_1^{-}(x,\lambda)]}{2i\lambda}, \ \lambda \in S_3 \\ B(\lambda) = \frac{[f_2^{-}(x,\lambda), f_1^{-}(x,\lambda)]}{2i\lambda}, \ \lambda \in S_2 \\ C(\lambda) = \frac{[f_1^{+}(x,\lambda), f_2^{+}(x,\lambda)]}{2i\lambda}, \ \lambda \in S_0 \\ D(\lambda) = \frac{[f_1^{+}(x,\lambda), f_2^{-}(x,\lambda)]}{2i\lambda}, \ \lambda \in S_1 \end{cases} \tag{18}$$

and

$$S_k = \left\{ \frac{k\pi}{2} < \arg\lambda < \frac{(k+1)\pi}{2} \right\}, \ k = \overline{0,3}.$$

Lemma 1. *The coefficients $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ have the asymptotics*

$$\begin{cases} A(\lambda) = \frac{1-i}{2} + O\left(\frac{1}{\lambda}\right), \ \lambda \in S_3 \\ B(\lambda) = \frac{1+i}{2} + O\left(\frac{1}{\lambda}\right), \ \lambda \in S_2 \\ C(\lambda) = \frac{1+i}{2} + O\left(\frac{1}{\lambda}\right), \ \lambda \in S_0 \\ D(\lambda) = \frac{1-i}{2} + O\left(\frac{1}{\lambda}\right), \ \lambda \in S_1 \end{cases} \tag{19}$$

Proof. Let us first prove it for the case $\lambda \in S_0$. Then it follows from formula (18) that

$$
\begin{aligned}
&C(\lambda) = \tfrac{1}{2i\lambda}[f_1^+(x,\lambda), f_2^+(x,\lambda)] = \\
&= \tfrac{1}{2i\lambda}[f_1^{+'}(0,\lambda) f_2^+(0,\lambda) - f_1^+(0,\lambda) f_2^{+'}(0,\lambda)] = \\
&= \tfrac{1}{2i\lambda}\{[1+\textstyle\int_{-\infty}^{0} K_2(0,t)e^{\lambda t}dt][i\lambda - K_1(0,0) + \textstyle\int_0^\infty K_{1_x}(0,t)e^{i\lambda t}dt] - \\
&-[1+\textstyle\int_0^\infty K_1(0,t)e^{i\lambda t}dt][\lambda + K_2(0,0) + \textstyle\int_{-\infty}^{0} K_{2_x}(0,t)e^{\lambda t}dt]\} = \\
&= \tfrac{1}{2i\lambda}\{i\lambda - K_1(0,0) + \textstyle\int_0^\infty K_{1_x}(0,t)e^{i\lambda t}dt + i\lambda\textstyle\int_{-\infty}^{0} K_2(0,t)e^{\lambda t}dt - \\
&-K_1(0,0)\textstyle\int_{-\infty}^{0} K_2(0,t)e^{\lambda t}dt + \left(\textstyle\int_0^\infty K_{1_x}(0,t)e^{i\lambda t}dt\right)\left(\textstyle\int_{-\infty}^{0} K_2(0,t)e^{\lambda t}dt\right) - \\
&-\lambda - K_2(0,0) - \textstyle\int_{-\infty}^{0} K_{2_x}(0,t)e^{\lambda t}dt - \lambda\textstyle\int_0^\infty K_1(0,t)e^{i\lambda t}dt - \\
&-K_2(0,0)\textstyle\int_0^\infty K_1(0,t)e^{i\lambda t}dt - \left(\textstyle\int_{-\infty}^{0} K_{2_x}(0,t)e^{\lambda t}dt\right)\left(\textstyle\int_0^\infty K_1(0,t)e^{i\lambda t}dt\right) = \\
&= \tfrac{1}{2i\lambda}\{\lambda(i-1) - [K_1(0,0) + K_2(0,0)] + \\
&+\textstyle\int_0^\infty K_{1_x}(0,t)e^{i\lambda t}dt - K_1(0,0)\textstyle\int_{-\infty}^{0} K_2(0,t)e^{\lambda t}dt + \\
&+\left(\textstyle\int_0^\infty K_{1_x}(0,t)e^{i\lambda t}dt\right)\left(\textstyle\int_{-\infty}^{0} K_2(0,t)e^{\lambda t}dt\right) - \\
&-\textstyle\int_{-\infty}^{0} K_{2_x}(0,t)e^{\lambda t}dt - K_2(0,0)\textstyle\int_0^\infty K_1(0,t)e^{i\lambda t}dt - \\
&-\left(\textstyle\int_0^\infty K_1(0,t)e^{i\lambda t}dt\right)\left(\textstyle\int_{-\infty}^{0} K_{2_x}(0,t)e^{\lambda t}dt\right) + iK_2(0,0) - \\
&-i\textstyle\int_{-\infty}^{0} K_{2_t}(0,t)e^{\lambda t}dt - iK_1(0,0) - i\textstyle\int_0^\infty K_{1_t}(0,t)e^{i\lambda t}dt\} = \\
&= \tfrac{1+i}{2} - \tfrac{1}{4i\lambda}\textstyle\int_{-\infty}^{\infty} q(t)\,dt - \tfrac{1}{4\lambda}\textstyle\int_{-\infty}^{0} q(t)\,dt + \tfrac{1}{4\lambda}\textstyle\int_0^\infty q(t)\,dt + O\left(\tfrac{1}{\lambda}\right)
\end{aligned}
$$

The remaining equations in formula (19) are proved in a very similar way.

3. The spectrum of the operator L

We first calculate the kernel of the resolvent of the operator $R(\lambda)$ to study the spectrum of the operator L generated by the differential expression (1). We prove the theorem from which we deduce the existence of the resolvent of the operator $R(\lambda)$.

We use the notation $\rho(L)$ to represent the resolvent set, $\sigma(L)$ to represent the spectrum, $\sigma_p(L)$ to denote the point spectrum, $\sigma_r(L)$ to denote the residual spectrum, and $\sigma_c(L)$ to denote the continuous spectrum of L.

Theorem 1. *The operator L has no purely real and purely imaginary eigenvalues.*

Proof. Eq. (4) has fundamental solutions

$f_1^+(x,\lambda), f_1^-(x,\lambda)\ \left(f_2^+(x,\lambda), f_2^-(x,\lambda)\right)$

if $\lambda \neq 0$, $\operatorname{Im}\lambda = 0$ ($\operatorname{Re}\lambda = 0$). It is possible to write a solution to Eq. (4) for $\operatorname{Im}\lambda = 0$ in the form of

$$
\begin{aligned}
y(x,\lambda) =& C_1\left(e^{i\operatorname{Re}\lambda x} + \int_x^\infty K_1(x,t)e^{i\operatorname{Re}\lambda t}dt\right) \\
&+ C_2\left(e^{-i\operatorname{Re}\lambda x} + \int_x^\infty K_1(x,t)e^{-i\operatorname{Re}\lambda t}dt\right).
\end{aligned}
$$

Since the principal parts of the solutions are periodic, $y(x,\lambda) \notin L_2(-\infty,\infty)$, for any non-zero values of C_1 and C_2. The case $\operatorname{Re}\lambda = 0$ can be proved analogously. Consequently, $\sigma_p(L) = \emptyset$. Theorem is proved.

Theorem 2. *Residual spectrum of the operator L is empty, $\sigma_r(L) = \emptyset$.*

Proof. The function $g(x) \in L_2(R)$, solution of $L^*(\lambda) = 0$ and $\overline{g(x)}$ satisfies

$$-g''(x,\lambda) + q(x)g(x,\lambda) = \lambda^2\rho(x)g(x,\lambda). \tag{20}$$

It is obvious that (20) cannot have a solution that belongs to $L_2(R)$, because it is of type (5). It means that $\sigma_p(L^*) = \emptyset$ or $\sigma_r(L) = \emptyset$ so $\sigma(L) = \sigma_c(L)$ and L^{-1} is defined in the dense set in $L_2(R)$ for $\forall\lambda \in C$.. The theorem is proved.

Using a general method it is possible to prove the following theorem

Theorem 3. *For each λ from the sector*

$$S_k = \left\{\frac{k\pi}{2} < \arg\lambda < \frac{(k+1)\pi}{2}\right\},\ k = \overline{0,3}$$

the resolvent of the operator $R(x,t,\lambda)$ exists and has the following form

$$
R(x,t,\lambda) = \begin{cases} R_0(x,t,\lambda), & \lambda \in S_0 \\ R_1(x,t,\lambda), & \lambda \in S_1 \\ R_2(x,t,\lambda), & \lambda \in S_2 \\ R_3(x,t,\lambda), & \lambda \in S_3 \end{cases}
$$

Here

$$
\begin{cases}
R_0(x,t,\lambda) = -\frac{1}{2i\lambda C(\lambda)}\begin{cases} f_1^+(x,\lambda)f_2^+(t,\lambda), & t<x \\ f_1^+(t,\lambda)f_2^+(x,\lambda), & t>x \end{cases};\ \lambda \in S_0 \\
R_1(x,t,\lambda) = -\frac{1}{2i\lambda D(\lambda)}\begin{cases} f_1^+(x,\lambda)f_2^-(t,\lambda), & t<x \\ f_1^+(t,\lambda)f_2^-(x,\lambda), & t>x \end{cases};\ \lambda \in S_1 \\
R_2(x,t,\lambda) = \frac{1}{2i\lambda B(\lambda)}\begin{cases} f_1^-(x,\lambda)f_2^-(t,\lambda), & t<x \\ f_1^-(t,\lambda)f_2^-(x,\lambda), & t>x \end{cases};\ \lambda \in S_2 \\
R_3(x,t,\lambda) = \frac{1}{2i\lambda A(\lambda)}\begin{cases} f_1^-(x,\lambda)f_2^+(t,\lambda), & t<x \\ f_1^-(t,\lambda)f_2^+(x,\lambda), & t>x \end{cases};\ \lambda \in S_3
\end{cases}
\tag{21}
$$

Lemma 2. *For every complex number $\lambda \notin \{\operatorname{Re}\lambda = 0\} \cup \{\operatorname{Im}\lambda = 0\}$, $A(\lambda) \neq 0$, $B(\lambda) \neq 0$, $C(\lambda) \neq 0$ and $D(\lambda) \neq 0$ there is a one-to-one resolvent operator $R_\lambda = \left(L - \lambda^2 I\right)^{-1}$.*

If we combine Theorems 1 and 2 we obtain the following result

Theorem 4. *The continuous spectra of the operator L fills out the axes*

$\{\operatorname{Re}\lambda = 0\} \cup \{\operatorname{Im}\lambda = 0\}$

Theorem 5. *The finite eigenvalues of the operator L coincide with the squares of the zeros of the functions $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ from the sectors S_k, $k = \overline{0,3}$.*

Corollary 1. *The functions $A(\lambda), B(\lambda), C(\lambda)$ and the function $D(\lambda)$ have no zeros on the axes*

$\{\operatorname{Re}\lambda = 0\} \cup \{\operatorname{Im}\lambda = 0\}$

Taking into account Corollary 1, by dividing both sides of the first and third equation of (17) by $C(\lambda)$ and the second and fourth by $B(\lambda)$ we obtain solutions for Eq. (4)

$$
\begin{cases}
U_1^+(x,\lambda) = \frac{D(\lambda)}{C(\lambda)}f_2^+(x,\lambda) - f_2^-(x,\lambda), & for\ \operatorname{Re}\lambda = 0 \\
U_1^-(x,\lambda) = \frac{A(\lambda)}{B(\lambda)}f_2^-(x,\lambda) - f_2^+(x,\lambda), & for\ \operatorname{Re}\lambda = 0 \\
U_2^+(x,\lambda) = \frac{A(\lambda)}{C(\lambda)}f_1^+(x,\lambda) + f_1^-(x,\lambda) & for\ \operatorname{Im}\lambda = 0 \\
U_2^-(x,\lambda) = \frac{D(\lambda)}{B(\lambda)}f_1^-(x,\lambda) + f_1^+(x,\lambda) & for\ \operatorname{Im}\lambda = 0
\end{cases}
\tag{22}
$$

The functions $[U_1^-, U_2^-]$ and functions $[U_1^+, U_2^+]$ are known as the eigenfunctions of the left and right spectral problem.

Definition 1. The functions

$$S_1^+(\lambda) = \frac{D(\lambda)}{C(\lambda)}, S_2^+(\lambda) = \frac{A(\lambda)}{C(\lambda)}, S_1^-(\lambda) = \frac{A(\lambda)}{B(\lambda)}, S_2^-(\lambda) = \frac{D(\lambda)}{B(\lambda)}$$

are called the reflection coefficients for Eq. (4).

4. Eigenfunction expansions

In Section 3 we proved that if $\operatorname{Im}\lambda > 0, \operatorname{Re}\lambda > 0$ then the kernel of the resolvent of the operator L has the form

$$R_0(x,t,\lambda) = \frac{1}{2i\lambda C(\lambda)}\begin{cases} f_1^+(x,\lambda)f_2^+(t,\lambda), & t<x \\ f_1^+(t,\lambda)f_2^+(x,\lambda), & t>x \end{cases};\ \lambda \in S_0.$$

Lemma 3. *For any twice differentiable continuous function $\psi(x) \in L_2(-\infty,\infty)$ and*

$g(x) = -\psi''(x) + q(x)\psi(x) \in L_2(-\infty,\infty)$

we have

$$\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{\psi(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\infty}^{\infty} R(x,t,\lambda)g(t)\,dt, \tag{23}$$

It is easy to show that if the conditions of the lemma are satisfied, then

$$\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{\psi(x)}{\lambda^2} + \frac{1}{\lambda^2} O\left(\frac{1}{\lambda^2}\right).$$

The left side of (23) is the analytic function for λ within the contour of the circle $|\lambda| = R$ except for the points $\lambda = \lambda_n$, $n = \overline{1,l}$. Then for $R \to \infty$ we have

$$\begin{aligned}\psi(x) = &-\tfrac{1}{2\pi i}\int_0^\infty 2\lambda d\lambda \int_{-\infty}^{\infty}[R_0(x,t,\lambda) - R_3(x,t,\lambda)]\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi i}\int_{-\infty}^0 2\lambda d\lambda \int_{-\infty}^{\infty}[R_1(x,t,\lambda) - R_2(x,t,\lambda)]\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi i}\int_0^{i\infty} 2\lambda d\lambda \int_{-\infty}^{\infty}[R_1(x,t,\lambda) - R_0(x,t,\lambda)]\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi i}\int_0^{-i\infty} 2\lambda d\lambda \int_{-\infty}^{\infty}[R_2(x,t,\lambda) - R_3(x,t,\lambda)]\rho(t)\psi(t)\,dt+\\ &+\sum_{n=1}^{l}\mathrm{Res}\,2\lambda\left(\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt\right)_{\lambda=\lambda_n}.\end{aligned} \tag{24}$$

Using formulas (17) and (21) we obtain

$$R_0(x,t,\lambda) - R_3(x,t,\lambda) = \frac{1}{2i\lambda A(\lambda)C(\lambda)} f_2^+(x,\lambda)\, f_2^+(t,\lambda)$$

$$R_1(x,t,\lambda) - R_0(x,t,\lambda) = \frac{1}{2i\lambda D(\lambda)C(\lambda)} f_1^+(x,\lambda)\, f_1^+(t,\lambda)$$

$$R_2(x,t,\lambda) - R_1(x,t,\lambda) = \frac{1}{2i\lambda D(\lambda)B(\lambda)} f_2^-(x,\lambda)\, f_2^-(t,\lambda)$$

$$R_3(x,t,\lambda) - R_2(x,t,\lambda) = \frac{1}{2i\lambda A(\lambda)B(\lambda)} f_1^-(x,\lambda)\, f_1^-(t,\lambda)$$

Then we have from (24)

$$\begin{aligned}\psi(x) = &\tfrac{1}{2\pi}\int_0^\infty \tfrac{1}{A(\lambda)C(\lambda)}d\lambda \int_{-\infty}^{\infty} f_2^+(x,\lambda)\, f_2^+(t,\lambda)\,\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi}\int_{-\infty}^0 \tfrac{1}{D(\lambda)B(\lambda)}d\lambda \int_{-\infty}^{\infty} f_2^-(x,\lambda)\, f_2^-(t,\lambda)\,\rho(t)\psi(t)\,dt-\\ &+\tfrac{1}{2\pi}\int_0^{i\infty} \tfrac{1}{D(\lambda)C(\lambda)}d\lambda \int_{-\infty}^{\infty} f_1^+(x,\lambda)\, f_1^+(t,\lambda)\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi i}\int_0^{-i\infty} \tfrac{1}{A(\lambda)B(\lambda)}d\lambda \int_{-\infty}^{\infty} f_1^-(x,\lambda)\, f_1^-(t,\lambda)\rho(t)\psi(t)\,dt+\\ &+\sum_{n=1}^{l}\mathrm{Res}\,2\lambda\left(\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt\right)_{\lambda=\lambda_n}.\end{aligned}$$

Thus the following theorem is proved

Theorem 6. *For each $\psi(x) \in L_2(-\infty,\infty)$ we have the following eigenfunction expansion is valid*

$$\begin{aligned}\psi(x) = &\tfrac{1}{2\pi}\int_0^\infty \tfrac{1}{A(\lambda)C(\lambda)}d\lambda \int_{-\infty}^{\infty} f_2^+(x,\lambda)\, f_2^+(t,\lambda)\,\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi}\int_{-\infty}^0 \tfrac{1}{D(\lambda)B(\lambda)}d\lambda \int_{-\infty}^{\infty} f_2^-(x,\lambda)\, f_2^-(t,\lambda)\,\rho(t)\psi(t)\,dt-\\ &+\tfrac{1}{2\pi}\int_0^{i\infty} \tfrac{1}{D(\lambda)C(\lambda)}d\lambda \int_{-\infty}^{\infty} f_1^+(x,\lambda)\, f_1^+(t,\lambda)\rho(t)\psi(t)\,dt-\\ &-\tfrac{1}{2\pi i}\int_0^{-i\infty} \tfrac{1}{A(\lambda)B(\lambda)}d\lambda \int_{-\infty}^{\infty} f_1^-(x,\lambda)\, f_1^-(t,\lambda)\rho(t)\psi(t)\,dt+\\ &+\sum_{n=1}^{l}\mathrm{Res}\,2\lambda\left(\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt\right)_{\lambda=\lambda_n}.\end{aligned} \tag{25}$$

If the eigenvalues of the function are simple, then

$$\begin{aligned}&\sum_{n=1}^{l}\mathrm{Res}\left(2\lambda\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt\right)_{\lambda=\lambda_n} =\\ &= \sum_{n=1}^{l} C_n \int_{-\infty}^{\infty} \tilde f_1(x,\lambda)\,\tilde f_1(t,\lambda)\rho(t)\psi(t)\,dt\end{aligned} \tag{26}$$

where

$$\tilde f_1(x,\lambda_n) = \begin{cases} f_1^+(x,\lambda_n), & \mathrm{Im}\,\lambda_n > 0\\ f_1^-(x,\lambda_n), & \mathrm{Im}\,\lambda_n < 0\end{cases}$$

and

$$C_n = i\begin{cases} \frac{1}{C_n^1}\underset{\lambda=\lambda_n}{\mathrm{Re}\,s}\,\frac{1}{C(\lambda)}, & \lambda_n \in S_0\\ \frac{1}{C_n^2}\underset{\lambda=\lambda_n}{\mathrm{Re}\,s}\,\frac{1}{D(\lambda)}, & \lambda_n \in S_1\\ \frac{1}{C_n^1}\underset{\lambda=\lambda_n}{\mathrm{Re}\,s}\,\frac{1}{B(\lambda)}, & \lambda_n \in S_2\\ \frac{1}{C_n^1}\underset{\lambda=\lambda_n}{\mathrm{Re}\,s}\,\frac{1}{A(\lambda)}, & \lambda_n \in S_3\end{cases}$$

Note that the formulas (25) and (26) can be written as follows

$$\begin{aligned}&-\tfrac{1}{2\pi}\int_0^\infty [S_2^+(\lambda) f_1^+(x,\lambda)\, f_1^+(t,\lambda) + \tfrac{1}{S_2^+(\lambda)} f_1^-(x,\lambda)\, f_1^-(t,\lambda)]d\lambda-\\ &-\tfrac{1}{2\pi}\int_{-\infty}^0 [S_2^-(\lambda) f_1^-(x,\lambda)\, f_1^-(t,\lambda) + \tfrac{1}{S_2^-(\lambda)} f_1^+(x,\lambda)\, f_1^+(t,\lambda)]d\lambda-\\ &-\tfrac{1}{2\pi}\int_{-\infty}^{\infty}[f_1^+(x,\lambda)\, f_1^-(t,\lambda) + f_1^-(x,\lambda)\, f_1^+(t,\lambda)]d\lambda+\\ &+\tfrac{i}{2\pi}\int_{-i\infty}^{i\infty}[f_2^+(x,\lambda)\, f_2^-(t,\lambda) + f_2^-(x,\lambda)\, f_2^+(t,\lambda)]d\lambda-\\ &-\tfrac{i}{2\pi}\int_{-i\infty}^0 [S_1^-(\lambda) f_2^-(x,\lambda)\, f_2^-(t,\lambda) + \tfrac{1}{S_1^-(\lambda)} f_2^+(x,\lambda)\, f_2^+(t,\lambda)]d\lambda-\\ &-\tfrac{i}{2\pi}\int_{-i\infty}^0 [S_1^+(\lambda) f_2^+(x,\lambda)\, f_2^+(t,\lambda) + \tfrac{1}{S_1^+(\lambda)} f_2^-(x,\lambda)\, f_2^-(t,\lambda)]d\lambda-\\ &-\sum_{n=1}^{l} C_n\int_{-\infty}^{\infty}\tilde f_1(x,\lambda)\,\tilde f_1(t,\lambda)\rho(t)\psi(t)\,dt = \delta(x-t)\end{aligned} \tag{27}$$

5. Inverse problem

In this section, using eigenvalue expansion (27) we provide the solution of the inverse scattering problem of recovering the potential $q(x)$ from the given scattering data $S_1^\pm(\lambda), S_2^\pm(\lambda)$.

First, let us consider the following integrals

$$\begin{aligned}A_1(x,t) &= \int_0^\infty [S_2^+(\lambda) f_1^+(x,\lambda) + f_1^-(x,\lambda)]e^{i\lambda t}d\lambda\\ A_2(x,t) &= \int_0^\infty [f_1^+(x,\lambda) + \tfrac{1}{S_2^+(\lambda)} f_1^-(x,\lambda)]e^{-i\lambda t}d\lambda\\ A_3(x,t) &= \int_{-\infty}^0 [f_1^-(x,\lambda) + \tfrac{1}{S_2^-(\lambda)} f_1^+(x,\lambda)]e^{i\lambda t}d\lambda\\ A_4(x,t) &= \int_{-\infty}^0 [S_2^-(\lambda) f_1^-(x,\lambda) + f_1^+(x,\lambda)]e^{-i\lambda t}d\lambda\\ A_5(x,t) &= i\int_{-\infty}^0 [S_1^-(i\lambda) f_1^-(x,\lambda) + f_1^+(x,\lambda)]e^{-i\lambda t}d\lambda\\ A_6(x,t) &= i\int_{-\infty}^0 [f_1^-(x,\lambda) + \tfrac{1}{S_1^-(i\lambda)} f_1^+(x,\lambda)]e^{-i\lambda t}d\lambda\\ A_7(x,t) &= i\int_0^\infty [S_1^+(i\lambda) f_1^+(x,\lambda) + f_1^-(x,\lambda)]e^{i\lambda t}d\lambda\\ A_8(x,t) &= i\int_0^\infty [f_1^+(x,\lambda) + \tfrac{1}{S_1^+(i\lambda)} f_1^-(x,\lambda)]e^{i\lambda t}d\lambda\end{aligned}$$

Using the formula (5) we obtain

$$\begin{aligned}A_1(x,t) = &\int_0^\infty \{S_2^+(\lambda)[e^{i\lambda x} + \int_x^\infty K_1(x,\xi)e^{i\lambda\xi}d\xi]+\\ &+\int_x^\infty K_1(x,\xi)e^{-i\lambda\xi}d\xi + e^{-i\lambda x}\}e^{i\lambda t}d\lambda =\\ &= \int_0^\infty [e^{-i\lambda(x-t)} - ie^{i\lambda(x+t)}]d\lambda + \int_0^\infty [S_2^+(\lambda) + i]e^{i\lambda(x+t)}d\lambda+\\ &+\int_x^\infty K_1(x,\xi)\int_0^\infty [S_2^+(\lambda) + i]e^{i\lambda(t+\xi)}d\lambda d\xi+\\ &+\int_x^\infty K_1(x,\xi)\int_0^\infty [e^{-i\lambda(\xi-t)} - ie^{i\lambda(\xi+t)}]d\lambda d\xi\end{aligned}$$

If we similarly calculate the remaining integrals and calculate the sum, we obtain the following expression:

$$-\frac{1}{2\pi}\sum_{n=1}^{8} A_n(x,t) = F_1(x+t) + \delta(x-t) + K_1(x,\xi) + \int_x^\infty K_1(x,t)F_1(t+\xi)\,d\xi, \tag{28}$$

where

$$\begin{aligned}-2\pi F_1(x+t) = &\int_0^\infty [\left(S_2^+(\lambda) + i\right)e^{i\lambda(x+t)} + \left(\tfrac{1}{S_2^+(\lambda)} - i\right)e^{-i\lambda(x+t)}]d\lambda+\\ &+\int_{-\infty}^0 [\left(S_2^-(\lambda) + i\right)e^{-i\lambda(x+t)} + \left(\tfrac{1}{S_2^-(\lambda)} - i\right)e^{i\lambda(x+t)}]d\lambda+\\ &+i\int_0^{i\infty} [\left(S_1^+(\lambda) + i\right)e^{\lambda(x+t)} + \left(\tfrac{1}{S_1^+(\lambda)} - i\right)e^{-\lambda(x+t)}]d\lambda\\ &+i\int_{-i\infty}^0 [\left(S_1^-(\lambda) + i\right)e^{-\lambda(x+t)} + \left(\tfrac{1}{S_1^-(\lambda)} - i\right)e^{\lambda(x+t)}]d\lambda.\end{aligned}$$

On the other hand

$$e^{\pm i\lambda t} = f_1^{\pm}(t,\lambda) + \int_t^{\infty} A(t,\xi) f_1^{\pm}(\xi,\lambda)\,d\xi.$$

This is the reason

$$\begin{aligned} A_1(x,t) &= \int_0^{\infty} [S_2^{+}(\lambda) f_1^{+}(x,\lambda) + f_1^{-}(x,\lambda)]e^{i\lambda t} d\lambda = \\ &= \int_0^{\infty} [S_2^{+}(\lambda) f_1^{+}(x,\lambda) + f_1^{-}(x,\lambda)][f_1^{+}(t,\lambda) + \int_t^{\infty} A(t,\xi) f_1^{+}(\xi,\lambda)\,d\xi]d\lambda = \\ &= \int_0^{\infty} [S_2^{+}(\lambda) f_1^{+}(x,\lambda) f_1^{+}(t,\lambda) + f_1^{+}(t,\lambda) f_1^{-}(x,\lambda)]d\lambda + \\ &+ \int_t^{\infty} A(t,\xi) \int_0^{\infty} [S_2^{+}(\lambda) f_1^{+}(x,\lambda) f_1^{+}(\xi,\lambda) + f_1^{+}(\xi,\lambda) f_1^{-}(x,\lambda)]d\lambda d\xi. \end{aligned}$$

If we calculate the remaining integral analogously and add the sum, we obtain that

$$-\frac{1}{2\pi}\sum_{n=1}^{8} A_n(x,t) = \delta(x-t) - \sum_{n=1}^{l} C_n f_1^{+}(x,\lambda_n)\,e^{i\lambda_n t} + A(t,x). \tag{29}$$

Since for $t > x$, $A(t,x) = 0$, we obtain by comparing formulas (27) and (28)

$$F(x+t) + K_1(x,t) + \int_x^{\infty} K_1(x,\xi) F(t+\xi)\,d\xi = 0 \tag{30}$$

where

$$F(x+t) = F_1(x+t) + \sum_{n=1}^{l} C_n e^{i\lambda_n(x+t)}$$

The following theorem can be proved in a similar way

Theorem 7. *For $t > x > 0$ the kernel*

$K_1(x,t)$

of representation (5) satisfies main Eq. (29).

Similarly, the following theorem can be proved

Theorem 8. *For $t < x < 0$ the kernel $K_2(x,t)$ of representation (6) satisfies to the main equation*

$$\tilde F(x+t) + K_2(x,t) + \int_{-\infty}^{x} K_2(x,\xi)\tilde F(t+\xi)\,d\xi = 0, \tag{31}$$

where

$$\begin{aligned} \tilde F(x+t) &= -\frac{i}{2\pi}\int_0^{\infty} [\left(S_2^{+}(\lambda) + i\right)e^{i\lambda(x+t)} + \left(\frac{1}{S_2^{+}(\lambda)} - i\right) e^{-i\lambda(x+t)}]d\lambda + \\ &-\frac{i}{2\pi}\int_{-\infty}^{0} [\left(S_2^{-}(\lambda) + i\right)e^{-i\lambda(x+t)} + \left(\frac{1}{S_2^{-}(\lambda)} - i\right) e^{i\lambda(x+t)}]d\lambda + \\ &-\frac{1}{2\pi}\int_0^{i\infty} [\left(S_1^{+}(\lambda) + i\right)e^{\lambda(x+t)} + \left(\frac{1}{S_1^{+}(\lambda)} - i\right) e^{-\lambda(x+t)}]d\lambda \\ &-\frac{1}{2\pi}\int_{-i\infty}^{0} [\left(S_1^{-}(\lambda) + i\right)e^{-\lambda(x+t)} + \left(\frac{1}{S_1^{-}(\lambda)} - i\right) e^{\lambda(x+t)}]d\lambda + \\ &+ \sum_{n=1}^{l} \tilde C_n e^{\lambda_n(x+t)} \end{aligned}$$

Lemma 1 and the formula (25) guarantee the convergence of these integrals.

6. Uniqueness

Theorem 9. *For any $x \in [0,\infty)$, main Eq. (30) has a unique solution belonging to $L_2[x,\infty)$*

Proof.

Taking into account the estimations

$$\int_0^{\infty} \left|F_1(x)\right| dx < \infty, \int_0^{\infty} (1+x)\left|\frac{d}{dx}F_1(x)\right| dx < \infty,$$
$$\int_{-\infty}^{0} \left|\tilde F_1(x)\right| dx < \infty, \int_{-\infty}^{0} (1-x)\left|\frac{d}{dx}\tilde F_1(x)\right| dx < \infty$$

we can state that the kernel $F_1(t+\xi)$ generates a completely continuous operator in space $L_2(x,\infty)$ for each $x \geq 0$. Therefore, it is enough to show that the homogeneous equation

$$f(t) + \int_x^{\infty} f(\xi) F(t+\xi)\,d\xi = 0 \tag{32}$$

has only a trivial solution in the space $L_2[x,\infty)$.

Let us denote by $Z_x(t)$ the solution of the Volterra integral equation

$$f(t) = Z_x(t) + \int_x^{t} K(\xi,t) Z_x(\xi)\,d\xi \tag{33}$$

Then from (32) we have

$$\begin{aligned} &Z_x(t) + \int_x^{t} Z_x(\xi)[F(t+\xi) + K(\xi,t) + \int_x^{\infty} F(t+u)K(\xi,u)\,du]d\xi + \\ &+ \int_t^{\infty} Z_x(\xi)[F(t+\xi) + K(\xi,t) + \int_\xi^{\infty} F(t+u)K(\xi,u)\,du]d\xi = 0 \end{aligned}$$

From the main Eq. (30) we find

$$Z_x(t) + \int_t^{\infty} Z_x(\xi)[F(t+\xi) + K(\xi,t) + \int_\xi^{\infty} F(t+u)K(\xi,u)\,du]d\xi = 0.$$

Taking into account estimates

$$\begin{aligned} &|F(t)| \le \int_t^{\infty} \left|F'(s)\right| ds = \alpha(t) \\ &|K(x,y)| \le |F(x+y)| + \int_x^{\infty} |K(x,t)|\,|F(t+y)|\,dt \le \\ &\le \alpha(x+y) + \alpha(x+y)\int_x^{\infty} |K(x,t)|dt \le C\alpha(x+y),\quad C - const\tan t \end{aligned}$$

and

$$\int_x^{\infty} |K(x,t)\,F(t+y)|dt \le |K(x,y)| + |F(x+y)| \le C_1\alpha(x+y)$$

in Eq. (39) we obtain that $Z_x(t) = 0$ for $t \geq x$. This implies $f(t) \equiv 0$. The theorem is proved.

Theorem 10. *For any $x \in (-\infty,0]$ main Eq. (30) has a unique solution belonging to $L_2(-\infty,x]$*

The Theorem 10 can be proved analogously.

7. Conclusions

In this paper, we discuss plane-wave propagation in the layered medium. For this purpose, we use complex potentials that have allowed for the modelling of neutron absorption where the imaginary potential signifies emission or absorption. Although there are various studies about the spectral analysis of these problems, many are for regular Sturm–Liouville boundary value problems. Moreover, our approach to investigating the eigenvalues and spectral singularities differs significantly from other papers. By using the integral representation of the Jost solutions on the real and imaginary axes, we solved the direct scattering problem. Integral equations play a crucial role in solving the inverse problem and ultimately prove the uniqueness theorem for the solution. It should be noted that a similar investigation can be done for using new differential operators, such as those presented in.[2,10,22]

Declaration of competing interest

The authors declare that they do not have any conflict of interests regarding submission and publication of this paper.

Data availability

No data was used for the research described in the article.

References

1. Naǐmark MA. Linear differential operators. (No Title). Nonselfadjoint operator. *Math USSR-Sbornik*. 1967;1(4):485–504.
2. Valdés Nápoles, Guzmán PM, Lugo LM, Kashurı A. The local generalized derivative and Mittag-Leffler function. *Sigma J Eng Nat Sci*. 2020;38(2):1007–1017.
3. Lyantse VE. An analog of the inverse problem of scattering theory for a nonselfadjoint operator. *Math USSR-Sbornik*. 1967;1(4):485–504.
4. Yurko V. Inverse problems for differential operators with indefinite discontinuous weights. *Results Math*. 2020;75(4):138.
5. Yurko V. An inverse problem for Sturm–Liouville operators on the half-line with complex weights. *J Inverse Ill-Posed Probl*. 2019;27(3):439–443.
6. Zhang R, Yang C-F, Bondarenko NP. Inverse spectral problems for the Dirac operator with complex-valued weight and discontinuity. *J Differential Equations*. 2021;278:100–110.

7. Ilhan OA, Manafian J. Periodic type and periodic cross-kink wave solutions to the (2+ 1)-dimensional breaking soliton equation arising in fluid dynamics. *Modern Phys Lett B*. 2019;33(23):1950277.
8. Li D, Manafian J, Ilhan OA, et al Solitary waves for the nonparaxial nonlinear Schrödinger equation. *Modern Phys Lett B*. 2024;38(01):2350204.
9. Khan S, Khan DF, Usman T, et al First-principles investigation of structural, electronic and thermoelectric properties of SmMg2X2 (X = P, As, Sb, Bi) zintl compounds. *Int J Mod Phys B*. 2024:2450437.
10. Nápoles Valdes JE. The non-integer local order calculus. *Phys Astron Int J*. 2023;7(3):163–168.
11. Belishev MI. Inverse problem of the scattering of plane waves for a class of layered media. *J Sov Math*. 1983;22(1):1014–1031. http://dx.doi.org/10.1007/bf01305284.
12. Agranovich ZS, Marchenko VA. *The Inverse Problem of Scattering Theory*. Courier Dover Publications; 2020 2020.
13. Faddeev LD. Properties of the S-matrix of the onedimensional schrodinger equation. *Tr Mat Inst Steklova*. 1964;73:314–336.
14. Efendiev RF, Orudzhev HD, El-Raheem ZF. Spectral analysis of wave propagation on branching strings. *Bound Value Probl*. 2016;2016:1–18.
15. Mamedov KR, Nabiev AA. Inverse problem of scattering theory for a class one-dimensional Schrödinger equation. *Quaest Math*. 2019;42(7):841–856.
16. Blashchak VA. An analogue of the inverse problem in the theory of scattering for a non-selfconjugate operator. I. *Differentsial'nye Uravneniya*. 1968;4(8):1519–1533.
17. Annaghili Sh M, et al On multiple eigenfunction expansion of an operator pencil with complex almost periodic potentials. *Stoch Model Comput Sci*. 2023;3(1).
18. Cebesoy S, Bairamov E, Aygar Y. Scattering problems of impulsive Schrödinger equations with matrix coefficients. *Ricerche Mat*. 2023;72(1):399–415.
19. Efendiev RF, Annaghili S. Inverse spectral problem of discontinuous non-self-adjoint operator pencil with almost periodic potentials. *Azerb J Math*. 2023;13(1).
20. Efendiev RF, Orudzhev HD, Bahlulzade SJ. Spectral analysis of the discontinuous Sturm–Liouville operator with almost-periodic potentials. *Adv Math Models Appl*. 2021;6(3):266–277.
21. Abd-El-Reheem ZF. On the scattering problem for the Sturm–Liouville equation on the half line with sign-valued weight coefficient. *Appl Anal*. 1995;57(3–4):333–339.
22. Castillo RE, Nápoles Valdés JE, Chaparro H. Omega derivative. *Gulf J Math*. 2024;16(1):55–67.