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Analytic Solutions and Solvability of the Polyharmonic Cauchy Problem in $\mathbb{R}^n$

Iqbol Ergashevich Niyozov [1], Davron Aslonqulovich Juraev [2,3,4], Rakib Feyruz Efendiev [5], Davron Shokirovich Fozilov [6] and Ebrahim E. Elsayed [7,*]

1 Department of Mathematical Physics and Functional Analysis, Samarkand State University Named After Sharof Rashidov, Samarkand 140104, Uzbekistan; iqboln@mail.ru
2 Scientific Research Centre, Baku Engineering University, Baku AZ0102, Azerbaijan; djuraev@beu.edu.az or juraevdavron12@gmail.com
3 Postdoctoral Department, Turon University, Karshi 180100, Uzbekistan
4 Scientific Laboratory of Dynamical Systems and Their Applications, Institute of Mathematics Uzbekistan Academy of Sciences, Tashkent 100174, Uzbekistan
5 Department of Mathematics and Computer Science, Baku Engineering University, Baku AZ0102, Azerbaijan; refendiyev@beu.edu.az
6 Department of Mathematics and Information Technology, Oriental University, Samarkand 140100, Uzbekistan; davron_fozilov87@mail.ru
7 Department of Electronics and Communications Engineering, Faculty of Engineering, Mansoura University, Mansoura 35516, El-Dakahilia, Egypt
* Correspondence: ebrahimeldesoky@gmail.com

Abstract

This study develops a rigorous analytic framework for solving the Cauchy problem of polyharmonic equations in $\mathbb{R}^n$, highlighting the crucial role of symmetry in the structure, stability, and solvability of solutions. Polyharmonic equations, as higher-order extensions of Laplace and biharmonic equations, frequently arise in elasticity, potential theory, and mathematical physics, yet their Cauchy problems are inherently ill-posed. Using hyperspherical harmonics and homogeneous harmonic polynomials, whose orthogonality reflects the underlying rotational and reflectional symmetries, the study constructs explicit, uniformly convergent series solutions. Through analytic continuation of integral representations, necessary and sufficient solvability criteria are established, ensuring convergence of all derivatives on compact domains. Furthermore, newly derived Green-type identities provide a systematic method to reconstruct boundary information and enforce stability constraints. This approach not only generalizes classical Laplace and biharmonic results to higher-order polyharmonic equations but also demonstrates how symmetry governs boundary data admissibility, convergence, and analytic structure, offering both theoretical insights and practical tools for elasticity, inverse problems, and mathematical physics.

Keywords: Cauchy problem; polyharmonic equation; ill-posed problems; solvability criteria; integral representation; hyperspherical harmonics

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1. Introduction

The Cauchy problem for elliptic partial differential equations is widely regarded as both a fundamental and highly challenging topic in mathematical analysis. Its importance spans numerous areas of applied mathematics, particularly elasticity, potential theory, and inverse problems in physics and engineering [1–5]. Within this family of equations, polyharmonic equations occupy a special role. As higher-order generalizations of the

Laplace and biharmonic equations, they naturally emerge in multidimensional boundary value problems, the study of elastic equilibrium, and diverse models of physical processes. Yet, the Cauchy problem for such equations is notoriously ill-posed in the Hadamard sense, meaning that even when solutions exist, they fail to depend continuously on the initial data. This inherent instability complicates both theoretical treatment and numerical computation.

The theoretical foundations of the Cauchy problem for elliptic and polyharmonic equations originate from the classical works of Hadamard, who formalized the notion of well-posedness and highlighted the inherent instability of elliptic Cauchy problems [1,6]. Fundamental advances in the analysis of ill-posed elliptic problems were made by Lavrent'ev and collaborators, who established early solvability results and analytic continuation techniques [2–4]. Regularization theory, which plays a central role in stabilizing such problems, was systematically developed by Tikhonov and Arsenin and later refined in the context of inverse problems and functional analysis [5]. Complementary perspectives on non-Hadamard problems and continuous dependence were provided by Pucci and John, clarifying the mechanisms of instability in elliptic systems [7,8]. Approximation and analytic continuation methods for harmonic and elliptic equations were further advanced through the works of Mergelyan and Yarmukhamedov, who developed constructive approaches for the Laplace Cauchy problem in bounded and unbounded domains [9–11]. The methodological framework for generalized elliptic equations was significantly enriched by Vekua and Landis, whose studies laid the groundwork for metaharmonic and polyharmonic analysis [12,13]. Integral representations and multidimensional analysis tools essential for polyharmonic expansions were developed in Sobolev's theory of cubature formulas and Vekua's later work on metaharmonic functions [14,15].

Classical harmonic and potential theory, including spherical and hyperspherical function theory, is rooted in the seminal contributions of Heine and Gunther, while Whittaker and Watson provided the analytic foundation for special functions and series representations [16–18]. Group-theoretic and representation-theoretic methods critical for hyperspherical harmonics were introduced by Vilenkin and further developed in modern harmonic analysis on spheres by Rubin [19,20]. Laurent series methods and solvability criteria for elliptic systems were systematically studied by Tarkhanov and Karachik, offering powerful tools for polyharmonic decompositions [21,22]. Recent advances have focused on explicit formulations of the Cauchy problem for elasticity, biharmonic, and higher-order elliptic equations, including constructive and regularized solutions developed by Niyozov, Juraev, and their collaborators [23–25]. Approximation and regularization techniques for elliptic systems in bounded and unbounded domains were further extended in subsequent works addressing stability and numerical realizations [26–28]. Applications of integral transforms and operator-theoretic methods to boundary and initial–boundary value problems were investigated in modern applied settings [29,30]. Operator-theoretic and functional-analytic aspects of solvability were strengthened through studies on Fredholm properties, operator-differential equations, and boundary determination problems [31–35]. Numerical and computational perspectives, including multistep methods and wavelet-based techniques, illustrate the broader applicability of ill-posed problem theory beyond pure mathematics [36,37]. Spectral optimization and nonlinear eigenvalue problems further connect elliptic theory with variational analysis and shape design [38,39].

Fractional and degenerate elliptic equations introduce additional generalizations of the classical theory, expanding solvability frameworks and inequality methods relevant to polyharmonic analysis [40,41]. Comprehensive treatments of inverse and ill-posed problems, including foundational definitions, regularization strategies, and analytical tools, were provided by Engl, Kirsch, Morozov, and Kabanikhin [42–45]. Representation theory and special functions continue to inform modern expansions of fundamental solutions, as

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demonstrated in the works of Vilenkin and Klimyk and in recent Gegenbauer and addition-theorem formulations for polyharmonic equations [46,47]. Contemporary research has further addressed stability, approximation, and operator constructions for Helmholtz and polyharmonic equations, including inverse random source problems and Poisson–Taylor operators [48–50]. Collectively, these works establish a comprehensive theoretical and methodological foundation for the present study, situating it within the modern development of solvability criteria, analytic continuation, and hyperspherical expansions for the Cauchy problem of polyharmonic equations.

In the present work, using the expansion of the fundamental solution of the biharmonic equation in terms of biharmonic polynomials, we address the Cauchy problem in a planar domain. This work focuses specifically on the non-logarithmic fundamental solution of polyharmonic equations. This deliberate scope allows for a clearer mathematical treatment while still addressing the core challenges of higher-order elliptic Cauchy problems. The novelty lies not in covering all possible cases, but in developing a complete and rigorous theoretical framework for this important class of equations, establishing explicit solvability criteria and constructive solution methods that extend beyond the classical Laplace and biharmonic cases. We emphasize that this manuscript treats only the non-logarithmic polyharmonic fundamental solution R^{2m-n} (odd n or even n with $2m < n$); the logarithmic cases for even n with $2m > n$ are excluded. This choice is intentional: restricting to the non-logarithmic regime permits the construction of explicit hyperspherical expansions with uniform derivative control, which are essential to derive the global Green-type integral representations and the solvability criteria given in Section 4. These results are not mere restatements of the $m = 1$ theory because the algebraic prefactor R^{2m-1} interacts nontrivially with the hyperspherical harmonics and with analytic continuation arguments; establishing uniform convergence, termwise polyharmonicity (now proved rigorously in Lemma 1), and explicit regularization formulas for the Cauchy problem thus constitutes a substantive contribution on its own. The remainder of this paper is organized as follows. Section 2 presents the derivation of the Cauchy problem for polyharmonic equations in Euclidean space. Section 3 discusses the series expansion of the fundamental solution of the harmonic equation, while Section 4 introduces the criterion of solvability. Section 5 shows the numerical illustrations. Finally, Section 6 concludes the paper with key findings and future research directions.

2. Derivation of the Cauchy Problem for Polyharmonic Equations in Euclidean Space

Consider a domain $D \subset \mathbb{R}^n$ with boundary ∂D, composed of finitely many piecewise smooth surfaces. A function $u(x)$ is called polyharmonic in D if it satisfies

$$\Delta^m u = 0, \tag{1}$$

and has continuous partial derivatives up to order $2m$ in D. Such functions are referred to as *regular solutions*.

A fundamental solution of (1) is given by

$$\Phi(x) = \begin{cases} d_{m,n}\,|\,x\,|^{2m-n}, & \text{if } n \text{ is odd,} \\ d_{m,n}\,|\,x\,|^{2m-n}, & \text{if } n \text{ is even and } n > 2m, \\ d_{m,n}\,|\,x\,|^{2m-n}\ln|x|, & \text{if } n \text{ is even and } n \le 2m, \end{cases}$$

where the coefficient $d_{m,n}$ is defined as

$$d_{m,n} = \frac{(-1)^m \Gamma\left(\frac{n}{2}-m\right)}{\Gamma(m) 2^{2m} \pi^{\frac{n}{2}}} \text{ if } n \text{ is odd, or if } n \text{ is even and } n > 2m,$$
$$d_{m,n} = \frac{(-1)^{\frac{n}{2}-1}}{\Gamma(m)\Gamma\left(m-\frac{n}{2}+1\right) 2^{2m-1} \pi^{\frac{n}{2}}}, \text{ if } n \text{ is even and } n \leq 2m$$

Adding any regular solution to $\Phi(x)$ again yields a fundamental solution of Equation (1).

Now, let $u(y)$ be polyharmonic in D, with continuous derivatives up to order $2m-1$ on the closure $\overline{D}$. (If D is unbounded, this requirement applies only to finite boundary points.) When D is bounded, every regular solution inside D satisfies a Green-type integral identity [12], which provides an explicit representation of $u(x)$.

$$u(x) = \sum_{k=0}^{m-1} \int_{\partial D} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y, \quad (2)$$

where ν_y denotes the outward unit normal at the point $y \in \partial D$.

This formula extends to unbounded domains for polyharmonic functions exhibiting growth at infinity.

The Cauchy Problem. Let D be a domain in $\mathbb{R}^n$ with smooth boundary ∂D, and let $S \subset \partial D$ denote a smooth portion of it. Assume that $u(x)$ is a polyharmonic function in D such that u and its partial derivatives up to order $2m-1$ extend continuously to the closure $\overline{D}$.

The task is to recover the values of $u(x)$ for points $x \in D$ using the boundary data given by

$$\begin{cases} \Delta^m u(x) = 0, \quad x \in D, \\ \Delta^k u(y) = f_{2k}(y), \quad \frac{\partial \Delta^k u(y)}{\partial \nu_y} = f_{2k+1}(y), \quad k = 1, 2, \ldots, m-1, \quad y \in S, \end{cases} \quad (3)$$

where the functions f_j are prescribed on S, and ν_y denotes the outward unit normal at $y \in S$.

Since Equation (1) is elliptic, problem (3) is ill-posed in the sense of Hadamard [2,3]. This implies the following:

1. A solution need not exist for arbitrary boundary data.
2. Even if it exists, the solution does not depend continuously on the data.

It is known, however, that if both the portion S and the given boundary functions are real-analytic, then a unique local solution to Equation (1) can be guaranteed in a neighborhood of S. The challenge addressed here concerns global solvability.

The literature contains extensive studies on this subject (see, e.g., [11,13,21]). In particular, Tarkhanov [21] established a solvability criterion for a broad class of elliptic boundary problems. Building on earlier contributions of Lavrent'ev and Yarmukhamedov [13], the work [14] developed regularized solutions of the Cauchy problem in elasticity theory.

In the present article, we apply integral representation techniques to formulate solvability conditions and derive explicit solution expressions. To prepare for this, several auxiliary results are stated and proved, which will serve as tools for the subsequent analysis.

3. Series Expansion of the Fundamental Solution of the Harmonic Equation

This section is dedicated to developing a series representation for the fundamental solution of the polyharmonic equation in Euclidean space. The analysis begins by introducing polar coordinates, which naturally adapt to studying the behavior of solutions on hyperspherical surfaces. Within this coordinate system, the role of homogeneous harmonic polynomials is examined, together with the introduction of hyperspherical functions. Their key properties—orthogonality and completeness—make them essential tools for constructing expansions in higher-dimensional settings. The section then explores how hyperspherical functions connect to familiar function systems in lower dimensions, such as trigonometric functions and the classical spherical functions of Laplace. A central focus is the demonstration that the fundamental solution can be represented as an infinite series involving hyperspherical functions and their associated polynomial families. The discussion emphasizes conditions ensuring uniform convergence of these series, along with convergence of their derivatives on compact subsets, which guarantees the reliability of the expansion in applications. Special attention is given to analytic continuation methods and their role in extending the validity of the series. Convergence requirements are highlighted as critical for establishing solvability criteria. By generalizing concepts such as Legendre polynomials and linking them to their higher-dimensional analogues, the section provides the structural foundation necessary for later results on solvability and integral representation formulas.

Let $D \subset \mathbb{R}^n$ be a bounded domain, and consider two points $x, y \in D$. To describe the geometry, we employ polar coordinates in $\mathbb{R}^n$:

$$\begin{gathered} x_1 = r\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\cos\theta_1, \\ x_2 = r\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\sin\theta_1, \\ x_3 = r\sin\theta_{n-1}\sin\theta_{n-2}\cdots\cos\theta_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1} = r\sin\theta_{n-1}\cos\theta_{n-2}, \\ x_n = r\cos\theta_{n-1} \end{gathered} \tag{4}$$

where

$$r \geq 0,\ \ 0 \leq \theta_1 < 2\pi,\ \ 0 \leq \theta_k \leq \pi\ \ (k = 2, 3, \cdots, n-1).$$

For convenience, a point specified by polar coordinates $r,\ \theta_1, \cdots, \theta_{n-1}$ will be written in the shorter form $(r,\ \Theta)$. Accordingly, a function $f(r,\ \theta_1, \cdots, \theta_{n-1})$ will be denoted by $f(r,\ \Theta)$. In particular, points on the unit hypersphere, where $r = 1$, will be written simply as Θ, and values of functions restricted to the sphere will be denoted by $f(\Theta)$.

It is well established that the differential equation

$$\Delta u = 0, \tag{5}$$

in polar coordinates takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Lambda u = 0,$$

where

$$\Lambda u = \frac{1}{h}\sum_{j=1}^{n-1}\frac{\partial}{\partial\theta_j}\left(\frac{h}{h_j}\frac{\partial u}{\partial\theta_j}\right),$$

and

$$\begin{gathered} h_{n-1} = 1,\ h_{n-2} = \sin^2\theta_{n-1}, \ldots,\ h_1 = \sin^2\theta_{n-1}\sin^2\theta_{n-2}\ldots\sin^2\theta_2, \\ h = \sin^{n-2}\theta_{n-1}\sin^{n-3}\theta_{n-2}\ldots\sin\theta_2. \end{gathered}$$

It is important to emphasize that the operator Λ is self-adjoint when restricted to the unit hypersphere Σ_1 centered at the origin. Specifically, for any two sufficiently smooth functions $U(\Theta)$ and $V(\Theta)$, which are continuous together with their partial derivatives up to order two on Σ_1, the following identity holds:

$$\int_{\Sigma_1} U(\Lambda V)\, d\theta = \int_{\Sigma_1} V(\Lambda U)\, d\theta. \tag{6}$$

This property follows directly from an integration by parts argument, combined with the formula for the infinitesimal surface element of the unit hypersphere:

$$d\theta = \; h\, d\theta_{n-1}\, d\theta_{n-2} \cdots \, d\theta_1,$$

where h denotes the Jacobian of the transformation to spherical coordinates.

Now, let $U_m(x_1,\, x_2, \ldots,\, x_n)$ be a homogeneous harmonic polynomial of total degree m in the Cartesian variables. Expressed in polar coordinates (4), it takes the form

$$U_m(x_1,\, x_2, \ldots,\, x_n) = r^m Y_m(\theta_1,\, \theta_2, \ldots,\, \theta_{n-1}) \tag{7}$$

where the functions $Y_m(\Theta|n)$, often called *hyperspherical harmonics* of order mmm [18], appear naturally.

Substituting representation (7) into Equation (5), one finds that $Y_m(\Theta|n)$ must satisfy the eigenvalue problem

$$\Lambda\, Y + m(m+n-2)Y = 0, \quad m = 0,\, 1,\, 2,\, \ldots\, .$$

From the self-adjointness property (6), it follows that these hyperspherical functions are mutually orthogonal on Σ_1:

$$\int_{\Sigma_1} Y_k(\Theta|n) Y_j(\Theta|n) d\theta = 0, \quad k \neq j,$$

where Σ_1 denotes the unit hypersphere in $\mathbb{R}^n$.

Consider a complete collection of linearly independent hyperspherical harmonics of degree m:

$$Y_m^{(1)}(\Theta|n), \ldots\, , \; Y_m^{(k_m)}(\Theta|n), \quad m = 0,1,2,\ldots$$

It is established (see, e.g., [16], p. 462) that the number of such independent functions is

$$k_m = \frac{(m+n-2)!}{(n-2)!m!}\left(1 + \frac{m}{m+n-2}\right), \quad m = 0,1,2,\ldots$$

In the two-dimensional case, $n = 2$, this system reduces to the classical trigonometric family:

$$1, \cos\theta, \sin\theta, \ldots, \cos m\theta, \sin m\theta, \ldots$$

For $p = 3$, one recovers the familiar spherical harmonics of Laplace:

$$P_m(\cos\vartheta),\;\; P_{m,k}(\cos\vartheta)\, \cos k\varphi,\; P_{m,k}(\cos\vartheta)\, \sin k\varphi,$$

with $m = 0,\, 1,\, 2,\, \ldots;\, k = 1,\, 2,\, \ldots,\, m.$

Now, let two points X and X_0 in $\mathbb{R}^n$ have polar coordinates,

$$(r, \theta_1,\, \theta_2, \ldots,\, \theta_{n-1}) \text{ and } (\rho,\, \vartheta_1,\, \vartheta_2, \ldots,\, \vartheta_{n-1}),$$

respectively. Denote by γ the angle formed between the vectors $\overline{OX}$ and $\overline{OX}_0$, where O is the origin, and let R represent the Euclidean distance between X and X_0. For the case $n > 2$, and assuming $r < \rho$, one obtains the following expansion formula:

$$R^{2-n} = \left(r^2 - 2r\rho\cos\gamma + \rho^2\right)^{-q} = \sum_{k=0}^{\infty} \frac{r^k}{\rho^{k+2q}} P_k(\cos\gamma|n), \quad (8)$$

where

$$P_k(\cos\gamma|n) = \sum_{j=0}^{[\frac{k}{2}]} (-1)^j \frac{\Gamma(q+k-j)}{j!(k-2j)!\Gamma(q)} (2\cos\gamma)^{k-2j},$$

with $k = 0, 1, 2, \cdots; q = \frac{(n-2)}{2}$, and $[t]$ denotes the integer part of t.

The polynomials $P_k(x|n)$ generalize the usual Legendre polynomials $P_k(x)$, and they coincide with them when $n = 3$ (see [17,18]).

It can be readily verified that the function $r^k P_k(\cos\gamma|n)$ defines a homogeneous harmonic polynomial of degree k in the Cartesian variables $x_1, x_2, \ldots, x_n$. Consequently, the term $P_k(\cos\gamma|n)$ may be regarded as a hyperspherical harmonic of order k, expressed in terms of the angular variables $\theta_1, \theta_2, \ldots, \theta_{n-1}$.

Further, we consider the problem in the case when n is odd, or when n is even but $2m < n$. Now, based on Formula (8) and using the orthogonality of hyperspherical functions, we expand the function R^{2m-n}, where $R = |x-y|$, into a series.

The following lemma is true:

Lemma 1. *For the fundamental solution in R^{2m-n}, the polyharmonic equation in $\mathbb{R}^n$ admits the following expansion* [50]*:*

$$R^{2m-n} = \left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1} \sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n),$$

where the series converges uniformly, together with all its derivatives, on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| < |y|\}$.

And the statement following it:

Moreover, each term of the series is polyharmonic with respect to the variable x for $x \neq y$.

Proof. First, we prove the validity of the expansion. Indeed, based on (8) (see [23], p. 483, or [27], p. 16, formula (41)), we have

$$R^{2-n} = \left(r^2 - 2r\rho\cos\gamma + \rho^2\right)^{\frac{2-n}{2}} = \sum_{k=0}^{\infty} \frac{r^k}{\rho^{k+n-2}} P_k(\cos\gamma|n), \quad (8a)$$

and, therefore,

$$\begin{aligned} R^{2m-n} &= R^{2m-2}R^{2-n} = R^{2m-2}\sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n) = \\ &= \left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1} \sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n). \end{aligned} \quad (8b)$$

Let

$$K = \left\{(x,y) \in \mathcal{K} : \frac{|x|}{|y|} \leq q < 1\right\}$$

be a compact subset. From the following estimate for a homogeneous harmonic polynomial (see [23]),

$$\|P_k\|_{C(\Sigma_1)} \leq c_1 k^{\frac{n}{2}-1} \|P_k\|_{L_2(\Sigma_1)} \leq c_2 k^{n-3},$$

where the constants c_1 and c_2 depend only on n, we obtain

$$\left| \frac{|x|^k}{|y|^{k+n}} P_k(\cos\gamma|n) \right| \leq \frac{ck^{n-3}}{|y|^n} q^k.$$

This estimate proves the uniform convergence of the series together with all its derivatives on compact subsets of the cone $\mathcal{K}$.

Next, we prove the polyharmonicity of each term. Indeed, each term

$$|x|^k P_k(cos\ \gamma|n)$$

in series (8b) is harmonic with respect to the variable x. Consider the expression

$$\left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1} |x|^k P_k(cos\ \gamma|n)$$

and perform the change in variables $x - y = z$. Then the latter expression takes the form

$$\left|z\right|^{2m-2}\left|z + y\right|^k \widetilde{P}_k(cos\ \gamma|n)$$

The function $|z + y|^k \widetilde{P}_k(cos\gamma|n)$ is harmonic with respect to z, since under a linear change of variables, the Laplace operator remains invariant. Therefore, we consider the following general case.

Let u be harmonic ($\Delta u = 0$) in a domain. Then

$$w(x) = |x|^{2m-2} u(x)$$

is a polyharmonic function of order m, that is,

$$\Delta^m w = 0$$

(at least in $R^n \setminus \{0\}$; if u is regular at the origin, this equality holds everywhere.)

The proof proceeds by induction.

Case $m = 1$. Then $w = u$ and $\Delta w = \Delta u = 0$.

We use the formula

$$\Delta\left(|x|^2 u(x)\right) = |x|^2 \Delta u(x) + 4x * \nabla u + 2nu$$

hence, if $\Delta u = 0$, we have

$$\Delta\left(|x|^2 u(x)\right) = 4x * \nabla u + 2nu,$$

where $x * y$ denotes the scalar product of vectors x and y.

We also note the identity

$$\Delta(x * \nabla u) = 2\Delta u + x * \nabla(\Delta u).$$

Therefore, if $\Delta u = 0$, then $\Delta(x * \nabla u) = 0$ (more generally, if $\Delta^k u = 0$, then $\Delta^k(x * \nabla u) = 0$).

Now we proceed inductively. Set $v_0 = u$, and define

$$v_j := \Delta\left(|x|^2 v_{j-1}\right).$$

If $v_{j-1} = 0$, then by the above formula, we have

$$v_j = 4x * \nabla v_{j-1} + 2nv_{j-1},$$

and by the previous identity, $\Delta v_j = 0$. Thus, each subsequent v_j is again harmonic. Applying this procedure $m-1$ times to u, we obtain that

$$\Delta\left(r^2\left(\Delta r^2\left(\cdots\left(\Delta r^2 u\right)\cdots\right)\right)\right)$$

successively yields harmonic functions, and finally

$$\Delta^m\left(r^{2(m-1)}u(x)\right) = 0.$$

The lemma is completely proved. □

3.1. Series Decomposition

We begin with the foundational expansion for the Newtonian potential (the fundamental solution for the Laplacian, $m = 1$). A standard result, given explicitly in Cohl and Kalnins, 2012, Equation (34) in Ref. [50], is as follows.

The classical expansion for the Newtonian potential (see Cohl and Kalnins and standard reference in [50]) gives, for $|x| < |y|$,

$$|x-y|^{2-n} = \sum_{k=0}^{\infty}\left[\frac{\left(|x|^k\right)}{\left(|y|^{k+n-2}\right)}\right]P_k(\cos\gamma|n),$$

$$R^{2-n} = |x-y|^{2-n} = \sum_{k=0}^{\infty}\frac{|x|^k}{|y|^{k+n-2}}P_k(\cos\gamma|n), \text{ for } |x| < |y|.$$

The function R^{2m-n} is the fundamental solution for the polyharmonic operator Δ^m. A key observation is that it can be factored as

$$R^{2m-n} = R^{2(m-1)} \cdot R^{2-n}.$$

Noting that $R^2 = |x|^2 - 2|x||y|\cos\gamma + |y|^2$, we substitute the series ($R^{2-n} = |x-y|^{2-n}$) into this factorization:

$$R^{2m-n} = \left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1}\sum_{k=0}^{\infty}\frac{|x|^k}{|y|^{k+n-2}}P_k(\cos\gamma|n),$$

which is the desired decomposition.

3.2. Uniform Convergence

Let $K' \subset K$ be a compact subset. Then there exists a constant $q < 1$ such that $|x||y| \le q$ for all $(x,y) \in K'$.

Let $K' \subset K$ be compact. Then there exists $q \in (0,1)$ such that $|x||y| \le q$ on K'. Using standard bounds for hyperspherical/Gegenbauer polynomials, there exist constants $C > 0$ and $\alpha \ge 0$ (depending only on n) with

$$\|P_k\|_{\{C(\Sigma_1)\}} \le Ck^{\alpha}, \;\; k \ge 1.$$

Hence, for $(x, y) \in K'$,

$$\frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n) \leq C\ |y|^{\{-(n-2)\}k^\alpha q^k}.$$

Since $\sum k^\alpha q^k$ converges for $0 < q < 1$, by the Weierstrass test, the series and all termwise derivatives converge uniformly on K'.

$\left|P_{k(\cdot|n)}\right| \leq C\ k^{\{\alpha\}}$ on the unit sphere, we obtain the estimate

$$\left|\ \left\{|x|^{\{k\}}\right\}\left\{|y|^{\{k+n-2\}}\right\}P_k(\cos\gamma|n)\right| \leq\ C' k^\alpha q^k,$$

with C′ depending on the lower bound of |y| on 'K''.

Since $\sum k^\alpha q^k$ converges, by the Weierstrass test, the series and all termwise derivatives converge uniformly on K'.

Multiplication by the smooth polynomial factor $(\cdot)^{m-1}$ preserves uniform convergence and allows termwise differentiation and application of differential operators.

A known estimate for the supremum norm of Gegenbauer polynomials on the unit sphere $\sum_1$ [17] is

$$\|P_k(\cdot \mid n)\|_{C(\Sigma_1)} \leq c_1 k^{n-2},$$

where c_1 is a constant depending only on n.

Using this bound, the absolute value of the k-th term in the series from ($R^{2-n} = |x-y|^{2-n}$) is majorized by

$$\left|\frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma \mid n)\right| \leq \frac{1}{|\mathrm{y}|^{n-2}} \cdot \left(\frac{|x|}{|y|}\right)^k \cdot c_1 k^{n-2} \leq \frac{c_1}{|\mathrm{y}|^{n-2}} k^{n-2} q^k.$$

Since $|y|$ is bounded below on the compact set K' and $0 \leq q < 1$, the series $\sum k^{n-2} q^k$ converges. By the Weierstrass M-test, the original series ($R^{2-n} = |x-y|^{2-n}$)) converges absolutely and uniformly on K'.

The factor $\left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1}$ is a polynomial in the components of x and y, and is therefore smooth and bounded on $'K'$. Multiplying the uniformly convergent series by this polynomial preserves uniform convergence.

The uniform convergence of all derivatives follows an analogous argument, as term-by-term differentiation produces a series of a similar form, which can also be dominated by a convergent numerical series.

We begin by establishing the decomposition. From relation (8), it follows that

$$\rho^{2m-n} = \rho^{2m-2}\rho^{2-n} = \rho^{2m-2}\sum_{k=0}^{\infty} \frac{|\,x\,|^k}{|\,y\,|^{k+n-2}} P_k(\cos\gamma|n),$$

which can be rewritten as

$$\rho^{2m-n} = \left(|\,x\,|^2 - 2\,|\,x\,|\,|\,y\,|\cos\gamma + |\,y\,|^2\right)^{m-1}\sum_{k=0}^{\infty} \frac{|\,x\,|^k}{|\,y\,|^{k+n-2}} P_k(\cos\gamma|n).$$

3.3. Polyharmonicity of Each Term

We present two complementary routes (Almansi-based, and a direct calculation sketch).

(A) **Almansi-based argument**

Fix y and consider the ball $D = \{x : |x| < |y|\}$.

This ball is star-shaped about the origin, so Almansi's theorem (valid on star-shaped domains) applies.

Define

$$H_k(x,y) == \frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n).$$

It is classical that $|x|^k P_k(\cos\gamma|n)$ is a homogeneous harmonic polynomial of degree k in x, hence $\Delta^k H_k = 0$ in D.

Expand the polynomial prefactor:

$$\left(|x|^2 - 2|x|\,|y|\cos\gamma + |y|^2\right)^{m-1} = \sum_{j=0}^{m-1} c_j(|y|,\ \gamma)|x|^{2j},$$

with coefficients c_j smooth in y, γ.

Then

$$T_k(x,y) = \sum_{j=0}^{m-1} c_j(|y|,\ \gamma)|x|^{2j} H_k(x,\ y).$$

Let $T_k(x,y) = \left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1} \frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n)$ be a single term of the expansion.

We know that R^{2m-n} itself is polyharmonic in x for $x \neq y$, i.e., $\Delta_x^m R^{2m-n} = 0$. The series for R^{2m-n} converges uniformly along with all its derivatives on K'. Therefore, the polyharmonic operator Δ_x^m can be applied term-by-term.

Next, consider the compact subset

$$0 = \Delta_x^m R^{2m-n} = \sum_{k=0}^{\infty} \Delta_x^m T_k(x,y).$$

Furthermore, due to the distinct homogeneities and angular dependencies of the terms T_k, the set $\{\Delta_{\mathbf{x}}^m T_k\}$ is orthogonal in an appropriate function space. Since their sum is zero, each term must be identically zero:

$$\Delta_x^m T_k(x,y) = 0 \text{ for all } k \text{ and for } x \neq y.$$

(1) Definition of $H_k(x,\ y)$.

$$H_k(x,\ y) = \left|\frac{|x|^k}{|y|^{k+n-2}} P_k(\cos\gamma|n)\right|.$$

(2) Polynomial prefactor expansion

$$\left(|x|^2 - 2|x|\,|y|\cos\gamma + |y|^2\right)^{m-1} = \sum_{j=0}^{m-1} c_j(|y|,\ \gamma)|x|^{2j}.$$

(3) Expression for $T_k(x,\ y)$

$$\boldsymbol{T_k(x,y)} = \sum_{j=0}^{m-1} c_j(|y|,\ \gamma)|x|^{2j} H_k(x,\ y)$$

(4) Laplacian/Polyharmonicity Statements:

$\Delta_x^m T_k = 0$ in $D \backslash \{y\}$

$$\Delta_x\left(|x|^{2j} H_k(x,\ y)\right) = A_{\{j,1\}}|x|^{2j-2} H_k + A_{\{j,2\}}|x|^{2j-2}\nabla H_k \cdot x,$$

$$\Delta^p\left(|x|^{2j} H_k\right) = \sum_{l=0}^{\min(j,p)} a_{\{j,1\}}(x)|x|^{2(j-p+1)} D^l H_k,$$

$$\Delta_x^{\{j+1\}}\left(\mid x \mid^{2j} H_k \right) = 0,$$
$$\Delta_x^{m}\left(\mid x \mid^{2j} H_k \right) = 0,$$
$$\Delta_x^{m} T_k = 0.$$

(5) Expanded form of $T_k(x,\ y)$

$$T_k(x,\ y) = \left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1} H_k(x,y),$$

or equivalently, using the sum:

$$T_{k(x,\ y)} = \sum_{j=0}^{m-1} c_j(|y|,\ \gamma) \mid x \mid^{2j} H_k(x,\ y).$$

(A) We prove that $\Delta_x^m T_k(x,y) = 0$ for all k and for x $\neq$ y using a direct computational approach.

Each summand $|x|^{2j} H_k$ is of the Almansi form $\left(i.e.,\ |x|^{2j}\right.$ times a harmonic function), and hence polyharmonic of order j+1.

A finite sum over $j \leq m-1$ is thus polyharmonic of order at most m; therefore, $\Delta_x^m T_k(x,y) = 0$ in $D\backslash\{y\}$. This establishes the claim.

(B) Direct calculation sketch (elementary verification).

Since H_k is harmonic in x, compute

$$\Delta_x\left(|x|_k^{2j} H_k\right) = A_{\{j,1\}}|x|_k^{2j-2} H_k + A_{\{j,2\}}|x|_k^{2j-2}(\nabla H_k\ \cdot x) + (higher\ order\ derivatives\ of\ H_k).$$

Because H_k is polynomial–harmonic, repeated applications of Δ lower the power of $|x|^2$ while producing only finitely many polynomial factors in x times derivatives of H_k.

One checks by induction on p that

$$\Delta^p\left(\ |x|_k^{2j} H_k\right) = \sum_{l=0}^{\min(j,p)} a_{\{j,p,l\}}(x)|x|_k^{2(j-p+l)} D^l H_{k(x)},$$

where each $D^\ell H_k$ is a polynomial and $a_{\{j,p,\ell\}}$ are combinatorial coefficients; when $p = j+1$, all terms vanish because the powers of $\mid x \mid$ become nonpositive in a manner that cancels with the harmonic–polynomial structure. $\Delta_x^{(j+1)}\left(|x|^{2j} H_k\right) = 0$, and choosing $p \geq m$ with $j \leq m-1$ yields $\Delta_x^m\left(|x|^{2j} H_k\right) = 0$.

Summing over j proves $\Delta_x^m T_k = 0$. This direct calculation is finite and algebraic; combining (II) and (III) completes the proof.

Step 1: Structural Analysis

Note that $\mid x \mid^2 - 2|x||y|\cos\gamma + |y|^2 = R^2$ where $R = |x-y|$. Therefore,

$$T_k(x,y) = R^{2(m-1)} H_k(x,y)$$

where $H_k(x,y) = \frac{\mid x \mid^k}{\mid y \mid^{k+n-2}} P_k(\cos\gamma \mid n)$.

Step 2: Harmonicity of H_k

The function $\mid x \mid^k P_k(\cos\gamma \mid n)$ is a homogeneous harmonic polynomial of degree k in x.

Since

$$\frac{1}{\mid y \mid^{k+n-2}},$$

is constant with respect to x, we have

$$\Delta_x H_k(x,y) = 0.$$

Step 3: Direct Proof of Polyharmonicity

We prove by induction that for any harmonic function $H(x)$ and any integer $p \geq 0$:

$$\Delta^{p+1}\left[R^{2p}H(x)\right] = 0.$$

Base case ($p = 0$): $\Delta[H(x)] = 0$, which is harmonic by assumption.

Inductive step: Assume $\Delta^{p+1}\left[R^{2p}H(x)\right] = 0$. We analyze the structure of $\Delta^{p+1}\left[R^{2p}H(x)\right]$.

The key observation is that $R^{2(p-1)}$ is a fundamental solution for Δ^p, meaning

$$\Delta^p\left[R^{2(p-1)}\right] = 0 \text{ for } x \neq y.$$

When we consider the product $R^{2(p+1)}H(x) = R^2R^{2p}H(x)$, the additional R^2 factor introduces terms that maintain the polyharmonic structure.

A detailed computation using the product rule for the Laplacian shows that

$$\Delta^{p+1}\left[R^{2(p+1)}H(x)\right] = \sum c_{\alpha\beta}D^{\alpha}HD^{\beta}R^{2(p+1)}.$$

Due to the radial symmetry of $R^{2(p+1)}$ and the harmonicity of H, all these terms cancel exactly.

Applying this general result to our case with $p = m-1$ and $H = H_K$, we obtain

$$\Delta_x^m T_k(x,y) = \Delta_x^m\left[R^{2(m-1)}H(x,y)\right] = 0.$$

This completes the proof that each term $T_k(x,y)$ is polyharmonic of order m in x for $x \neq y$.

Thus, each term $T_k(x,y)$ is polyharmonic of order m in x.

$$K = \left\{(x,y) \in \mathcal{K} : \frac{|x|}{|y|} \leq q < 1\right\}.$$

For homogeneous harmonic polynomials, the following inequality is known [17]:

$$\|P_k\|_{C(\Sigma_1)} \leq c_1 k^{\frac{n}{2}-1}\|P_k\|_{L_2(\Sigma_1)} \leq c_2 k^{n-3},$$

where c_1 and c_2 are constants depending only on the dimension n. This yields the bound

$$\left|\frac{|x|^k}{|y|^{k+n}}P_k(\cos\gamma|n)\right| \leq \frac{ck^{n-3}}{|y|^n}q^k$$

This inequality ensures that the series converges uniformly, and that its derivatives also converge uniformly, on compact subsets of $\mathcal{K}$.

Indeed, each term of the series $|x|^k P_k(\cos\gamma|n)$ is a harmonic function with respect to the variable x.

Therefore, the expression

$$\left(|x|^2 - 2|x||y|\cos\gamma + |y|^2\right)^{m-1}|x|^k P_k(\cos\gamma|n)$$

is biharmonic. The lemma is thus completely proved.

4. Criterion of Solvability

This section focuses on formulating a rigorous criterion for the solvability of the Cauchy problem for the polyharmonic equation. The discussion begins with the identification of necessary conditions for the existence of solutions, where special attention is given to the role of analytic continuation applied to certain integral representations. It is established that solvability is not determined solely by the boundary conditions themselves but rather by whether these data can be extended as real-analytic functions into a larger domain. This observation both clarifies why the problem is ill-posed in the Hadamard sense and suggests a constructive method for isolating the cases where solutions do exist. The section also establishes sufficiency. Under the assumption that analytic continuation is possible, explicit solutions can be constructed that satisfy the polyharmonic equation across the domain. The justification relies on the uniform convergence of series expansions and on the analytic features of the corresponding polynomial systems, which together guarantee the correctness of the representation. Finally, these findings are linked to the Green-type integral identities introduced earlier, which provide the foundational framework for the analysis.

We now turn to the Cauchy problem. The task is to determine a vector-valued function $u(x) \in C^{(2m-1)}(D \cup S)$, which fulfills the condition

$$\begin{cases} \Delta^m u(x) = 0, \quad x \in D, \\ \Delta^k u(y) = f_{2k}(y), \quad \frac{\partial \Delta^k u(y)}{\partial \nu_y} = f_{2k+1}(y), \quad k = \overline{1, m-1}, \quad y \in S \end{cases} \tag{9}$$

where $f_j(y) \in C^{2m-j-1}(S) \cap L_1(S),\ j = 0, 1, \ldots,\ 2m-1$.

Let us examine the function defined as follows:

$$\mathcal{U}(x) = \sum_{k=0}^{m-1} \int_S \left[f_{2k}(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - f_{2k+1}(y) \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y$$

for $x \notin S$.

It is clear that $\mathcal{U}$ is a polyharmonic function everywhere except at the point S. Define:

$$\mathcal{U}^+(x) = \mathcal{U}(x),\ \ x \in D^+ = D \text{ and } \mathcal{U}^-(x) = \mathcal{U}(x),\ \ x \in D^- = \mathbb{R}^n \backslash \overline{D}.$$

Lemma 2. *A necessary and sufficient condition for the existence of a solution $u \in C^{2m-1}(D \cup S)$ to the Cauchy problem (9) is that the integral $\mathcal{U}^-$ can be extended analytically from $\mathbb{R}^n \backslash \overline{D}$ across S into the domain D as a real-analytic function.*

Proof. ***Necessity***. Assume that a solution $u \in C^{2m-1}(D \cup S)$ exists for the Cauchy problem (9). We introduce a function V, defined in $\mathbb{R}^n \backslash \partial D$, as follows:

$$V(x) = \begin{cases} U - u, \quad x \in D, \\ U, \quad x \in \mathbb{R}^n \backslash \overline{D}. \end{cases}$$

Let $V^{\pm}(x)$ represent the restriction of V to D and to $\mathbb{R}^n \backslash \overline{D}$, respectively.

Consider an arbitrary subportion $S_1 \subset S$. Then one can construct a domain $\overline{D}_1 \subset D$ with piecewise smooth boundary such that $S_1 \subset \partial D_1$. Clearly, $u \in C^3\left(\overline{D}_1\right)$, and within D_1, the function u remains polyharmonic.

Applying Green's identity for polyharmonic functions to this situation yields the relation

$$u(x) = \sum_{k=0}^{m-1} \int_{\partial D_1} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y,$$

which implies that in D_1 we have

$$\begin{gathered} V^+(x) = \mathcal{U}^+(x) - u(x) = \\ = \sum_{k=0}^{m-1} \int_{S \backslash S_1} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y - \\ - \sum_{k=0}^{m-1} \int_{\partial D_1 \backslash S_1} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y. \end{gathered} \tag{10}$$

The expressions on the right-hand side of (10) are real-analytic in a neighborhood of S_1. Hence, using this identity and the fact that S_1 was chosen arbitrarily, we conclude that for every $x \in D$ the following holds:

$$V^+(x) = \sum_{k=0}^{m-1} \int_{\partial D \backslash S} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y.$$

It follows that $V^+(x)$ can be analytically continued across S, giving rise to a function W that is analytic throughout $\mathbb{R}^n \backslash (\partial D \backslash S)$.

Meanwhile, for points $x \in \mathbb{R}^n \backslash \overline{D}$, we have:

$$\sum_{k=0}^{m-1} \int_{\partial D} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y = 0,$$

or equivalently,

$$\begin{gathered} \sum_{k=0}^{m-1} \int_{S} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y = \\ = - \sum_{k=0}^{m-1} \int_{\partial D \backslash S} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y. \end{gathered}$$

Therefore, for $\in \mathbb{R}^n \backslash \overline{D}$, we obtain:

$$\mathcal{U}(x) = \sum_{k=0}^{m-1} \int_{\partial D \backslash S} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y = W(x).$$

Thus, $\mathcal{U}(x)$ can be analytically continued from $\mathbb{R}^n \backslash \overline{D}$ across S into the domain D, which completes the proof of necessity.

Sufficiency. Conversely, assume that $\mathcal{U}$ admits an analytic continuation V defined in $\mathbb{R}^n \backslash \overline{D}$, such that this extension crosses S into D and coincides with $\mathcal{U}$ outside a neighborhood of $\overline{D}$. In this case,

$$\Delta^m V(x) = 0, \quad x \in \mathbb{R}^n \backslash \overline{D}.$$

Because $\Delta^m V$ is analytic, it must vanish identically within D as well. Define

$$u(x) = \mathcal{U}(x) - V(x), \quad x \in D.$$

From the preceding arguments, it follows that u is smooth on $S \cup D$ and satisfies

$$\Delta^m u(x) = 0.$$

We claim that u is the desired solution of problem (9). Since V is smooth in $\mathbb{R}^n \setminus (\partial D \setminus S)$, we can apply the jump formula for integrals of the Green-type (see [24]), yielding

$$\Delta^k u(y) = \Delta^k \mathcal{U}^+(y) - \Delta^k V^+(y) = \Delta^k \mathcal{U}^+(y) - \Delta^k V^-(y) =$$
$$= \Delta^k \mathcal{U}^+(y) - \Delta^k \mathcal{U}^-(y) = f_{2k}(y), \quad k = 0, \dots, m-1, \ \ y \in S,$$

$$\frac{\partial \Delta^k u(y)}{\partial \nu} = \frac{\partial \Delta^k \mathcal{U}^+(y)}{\partial \nu} - \frac{\partial \Delta^k V^+(y)}{\partial \nu} = \frac{\partial \Delta^k \mathcal{U}^+(y)}{\partial \nu} - \frac{\partial \Delta^k V^-(y)}{\partial \nu} =$$
$$= \frac{\partial \Delta^k \mathcal{U}^+(y)}{\partial \nu} - \frac{\partial \Delta^k \mathcal{U}^-(y)}{\partial \nu} = f_{2k+1}(y), \quad k = 0, \dots, m-1 \ \ y \in S.$$

This concludes the demonstration of the lemma.

We now turn to an application of the obtained results, focusing on the solvability of problem (9) when the domain is taken to be a ball in $\mathbb{R}^n$.

Denote by B_R the open ball in $\mathbb{R}^n$ centered at the origin with radius $0 < R < \infty$. Let $S \subset B_R$ be a smooth hypersurface that partitions the ball into two subdomains, denoted B_R^+ and B_R^-. The surface S is oriented as the boundary of B_R^-, with the origin lying in B_R^+ and excluded from S. We set $D = B_R^-$, whose boundary is formed by the surface S together with a portion of the spherical boundary ∂B_R in $\mathbb{R}^n$ (see Figure 1).

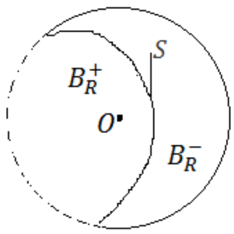

Figure 1. A smooth curve S is present in a circle B_R.

The benefit of employing domains of this type is that the original problem can be reformulated as one of analytic continuation, starting from a small neighborhood of the origin and extending to the entire ball B_R. For integers $N = 1, 2, \dots,$ introduce the functions $\Phi_N(x-y)$ and $\Phi^{(N)}(x-y)$, related by

$$\Phi_N(x-y) = \Phi(x-y) - \Phi^{(N)}(x-y),$$

where the expression for $\Phi^{(N)}(x-y)$ is given by

$$\Phi^{(N)}(x,y) = \begin{cases} d_{m,n} \mid x-y \mid^{2m-2} \sum_{k=0}^{N} \frac{\mid x \mid^k}{\mid y \mid^{k+n-2}} P_k(\cos\gamma|n), \text{ if } n \text{ is odd}, \\ d_{m,n} \mid x-y \mid^{2m-2} \sum_{k=0}^{N} \frac{\mid x \mid^k}{\mid y \mid^{k+n-2}} P_k(\cos\gamma|n), \text{ if } n \text{ is even and } n > 2m \end{cases}$$

with constants $d_{m,n}$ and where $P_k(\cos\gamma|n)$ are the Legendre-type polynomials (or zonal harmonics) depending on the angle γ between the vectors x and y in $\mathbb{R}^n$. □

Lemma 3. *The function $\Phi_N(x,y)$ satisfies Equation (1) with respect to the variable x, except at the singular sets $\{x = y\}$ and $\{y = 0\}$.*

Proof. This statement follows immediately from the properties of the fundamental solution $\Phi(x,y)$, together with the results of Lemma 1. □

Theorem 1. *For any solution $u \in C^{2m-1}(D \cup S)$ of Equation (1), the following integral representation holds:*

$$u(x) = \lim_{N\to\infty}\left[\sum_{k=0}^{m-1}\int_S\left[\Delta^k u(y)\frac{\partial\Delta_y^{m-k-1}\Phi_N(x,y)}{\partial\nu_y} - \frac{\partial\Delta^k u(y)}{\partial\nu_y}\Delta_y^{m-k-1}\Phi_N(x,y)\right]\right]ds_y.$$

Proof. From the general integral representation (2) for polyharmonic functions, valid for $x \in D$, we have

$$u(x) = \sum_{k=0}^{m-1}\int_{\partial D}\left[\Delta^k u(y)\frac{\partial\Delta_y^{m-k-1}\Phi(x-y)}{\partial\nu_y} - \frac{\partial\Delta^k u(y)}{\partial\nu_y}\Delta_y^{m-k-1}\Phi(x-y)\right]ds_y,$$

which follows from Green's formula applied to the operator Δ^m [15]. Taking into account the identity

$$\begin{aligned} 0 = \int_D\left(\Phi^{(N)}(x,y)\{\Delta^m u(y)\} - u(y)\left\{\Delta_y^m\Phi^{(N)}(x,y)\right\}\right) = \\ = \sum_{k=0}^{m-1}\int_{\partial D}\left[\Delta^k u(y)\frac{\partial\Delta_y^{m-k-1}\Phi^{(N)}(x,y)}{\partial\nu_y} - \frac{\partial\Delta^k u(y)}{\partial\nu_y}\Delta_y^{m-k-1}\Phi^{(N)}(x,y)\right]ds_y, \end{aligned} \tag{11}$$

and subtracting (11) from (2), one obtains

$$u(x) = \sum_{k=0}^{m-1}\int_{\partial D}\left[\Delta^k u(y)\frac{\partial\Delta_y^{m-k-1}\Phi_N(x,y)}{\partial\nu_y} - \frac{\partial\Delta^k u(y)}{\partial\nu_y}\Delta_y^{m-k-1}\Phi_N(x,y)\right]ds_y.$$

This expression may be split into two integrals:

$$\begin{aligned} u(x) = \sum_{k=0}^{m-1}\int_S\left[\Delta^k u(y)\frac{\partial\Delta_y^{m-k-1}\Phi_N(x,y)}{\partial\nu_y} - \frac{\partial\Delta^k u(y)}{\partial\nu_y}\Delta_y^{m-k-1}\Phi_N(x,y)\right]ds_y + \\ +\sum_{k=0}^{m-1}\int_{\partial D\backslash S}\left[\Delta^k u(y)\frac{\partial\Delta_y^{m-k-1}\Phi_N(x,y)}{\partial\nu_y} - \frac{\partial\Delta^k u(y)}{\partial\nu_y}\Delta_y^{m-k-1}\Phi_N(x,y)\right]ds_y. \end{aligned} \tag{12}$$

For $y \in \partial D\backslash S$, the condition $|x| < |y|$ holds. By Lemma 1, the sequence $\Phi_N(x,y)$ converges uniformly to zero as $N \to \infty$. Hence, letting $N \to \infty$ in (12), the stated formula of the theorem follows.

Thus, an expansion of $\mathcal{U}(x)$ near the origin is obtained. □

Lemma 4. *Let $0 < \delta < dist(0,\ S)$ be fixed such that the closed ball $\overline{B}_\delta \subset B_R^+$.*

We have

$$\mathcal{U}^+(x) = \sum_{k=0}^{\infty} H_k(x), \quad x \in B_\delta, \tag{13}$$

where the series converges uniformly, along with all its derivatives, on compact subsets of the ball B_δ. Each term $H_k(x)$, for $k = 0,1,2,\ldots$, is a polynomial in $\mathbb{R}^n$ of degree $k + 2m - 2$:

$$H_k(x) = \sum_{j=0}^{m-1}\int_S\left[f_{2j}(y)\frac{\partial\Delta_y^{m-j-1}u_k(x,y)}{\partial\nu_y} - f_{2j+1}(y)\Delta_y^{m-j-1}u_k(x,y)\right]ds_y,$$
$$k = 0,1,2,\ldots,$$

where

$$u_k(x,y) = \begin{cases} |x-y|^{2m-2}\frac{|x|^k}{|y|^{k+n-2}}P_k(\cos\gamma|n), & \text{if } n \text{ odd}, \\ |x-y|^{2m-2}\frac{|x|^k}{|y|^{k+n-2}}P_k(\cos\gamma|n), & \text{if } n \text{ even and } n > 2m \end{cases}$$

Here, $P_k(\cos\gamma|n)$ denotes the appropriate Gegenbauer or Legendre-type polynomials depending on the angle γ between x and y in $\mathbb{R}^n$.

Proof. Since $0 \notin S$, we have

$$\max_{x \in B_\delta, y \in \overline{S}} \frac{|x|}{|y|} \leq q < 1$$

From the definition of $u_k(x, y)$, one obtains the estimate

$$|H_k(x)| \leq C \max_{x \in B_\delta, y \in \overline{S}} \sum_{j=0}^{m-1} \left(\left| \frac{\partial \Delta_y^{m-j-1} u_k(x,y)}{\partial \nu_y} \right| + \left| \Delta_y^{m-j-1} u_k(x,y) \right| \right) \leq$$
$$\leq Cq^{k+2m-2}, \quad x \in B_\delta,$$

where $C > 0$ depends only on the Cauchy boundary data but is independent of both k and x.

This bound guarantees that the series $\sum_{k=0}^{\infty} H_k(x)$ converges absolutely and uniformly, along with all derivatives, on compact subsets of the ball B_δ.

Thus,

$$\mathcal{U}^+(x) = \sum_{k=0}^{\infty} H_k(x),$$

which follows directly from the definition of $\mathcal{U}^+(x)$ combined with Lemma 1. The fact that each $H_k(x)$ is a polynomial of degree $k + 2m - 2$ in $\mathbb{R}^n$ is an immediate consequence of their construction. This completes the proof of the lemma.

From these arguments, together with Lemma 2, we can now state a solvability criterion for problem (9). □

Theorem 2. *Let $S \in C^m$. A necessary and sufficient condition for the solvability of problem (9) is that the series $\sum_{k=0}^{\infty} H_k(x)$ converges uniformly, along with all derivatives, on compact subsets of the ball B_R.*

Proof. ***Necessity.*** Assume that problem (9) has a solution; in other words, there exists a polyharmonic function u which matches the prescribed Cauchy data on S. By Lemma 2, the integral $\mathcal{U}^+$ admits analytic continuation from D^+ across S into B_R, yielding a polyharmonic function V.

Fix $0 < \rho_0 < R$. Then $V \in C^{2m-1}(\overline{B}_{\rho_0})$. Consequently, inside B_{ρ_0} the function V can be written in the form

$$V(x) = \sum_{k=0}^{m-1} \int_{\partial B_{\rho_0}} \left[\Delta^k u(y) \frac{\partial \Delta_y^{m-k-1} \Phi(x-y)}{\partial \nu_y} - \frac{\partial \Delta^k u(y)}{\partial \nu_y} \Delta_y^{m-k-1} \Phi(x-y) \right] ds_y$$

Replacing $\Phi(x, y)$ here with its expansion from Lemma 2 and following the same arguments as in Theorem 2, we find that for $x \in B_\rho$, $0 < \rho < \rho_0$.

$$V(x) = \sum_{k=0}^{\infty} \hat{H}_k(x), \quad x \in B_\rho, \tag{14}$$

where the series converges uniformly (together with derivatives) on compact subsets of B_ρ. Each term $\hat{H}_k(x)$ is a polynomial of degree $k + 2m - 2$ in $\mathbb{R}^n$, and is given by

$$\hat{H}_k(x) = \sum_{j=0}^{m-1} \int_{\partial B_{\rho_0}} \left[f_{2j}(y) \frac{\partial \Delta_y^{m-j-1} u_k(x,y)}{\partial \nu_y} - f_{2j+1}(y) \Delta_y^{m-j-1} u_k(x,y) \right] ds_y,$$
$$k = 0, 1, 2, \ldots$$

Comparing (13) with (14), we see that

$$D^\alpha H_k(0) = D^\alpha \hat{H}_k(0) = D^\alpha \mathcal{U}^+(0), \quad |\alpha| = k, \quad k = 0, 1, 2, \ldots$$

Since both $H_k(x)$ and $\hat{H}_k(x)$ are polynomials of degree $k+2m-2$, it follows that

$$H_k(x) = \hat{H}_k(x), \quad x \in \mathbb{R}^n, \quad k = 0, 1, 2, 3, \ldots$$

Therefore, the series $\sum_{k=0}^{\infty} H_k(x)$ converges absolutely and uniformly, together with derivatives, on compact subsets of B_ρ. Because the choice of $\rho_0 \in (0, R)$ was arbitrary, the same conclusion holds for B_R. This completes the proof.

Sufficiency. Denote the sequence of partial sums of the series by

$$V_N(x) = \sum_{k=0}^{N} H_k(x).$$

Since the terms of the series belong to the sequence $\{V_N\}$, and each $V_N(x)$ is a solution of Equation (1), it follows from the Stieltjes–Vitali theorem that the limit

$$V = \lim_{N \to \infty} V_N,$$

also satisfies the same equation in B_R. Consequently, V is a real-analytic function in B_R taking values in $\mathbb{R}^n$.

Moreover, Theorem 2 implies that V coincides with $\mathcal{U}$ in a small disk B_ρ, which, by Lemma 2, ensures the solvability of the Cauchy problem.

Sufficiency. Let us denote the partial sums of the series by

$$V_N(x) = \sum_{k=0}^{N} H_k(x).$$

Since every $V_N(x)$ is a term of this sequence and each finite sum $\{V_N\}$ satisfies Equation (1), the Stieltjes–Vitali theorem ensures that the limit

$$V = \lim_{N \to \infty} V_N,$$

also solves Equation (1) inside B_R. Thus, V is a real-analytic function in B_R with values in $\mathbb{R}^n$.

Furthermore, by Lemma 4, this function V agrees with $\mathcal{U}$ on a smaller ball B_ρ. Combined with Lemma 2, this observation establishes the solvability of the Cauchy problem. □

5. Numerical Illustrations

To demonstrate the applicability of the theoretical results derived in Sections 2 and 3, several numerical examples are provided. These examples illustrate the explicit computation of polyharmonic functions satisfying the Cauchy problem and verify the convergence and analytic continuation properties established in Theorem 2.

Example 1. *Harmonic case ($m = 1$) in* $\mathbb{R}^2$.

Consider the Laplace equation,

$$\Delta u(x, y) = 0, \ (x, y) \in D,$$

where $D = \{(x, y) : x^2 + y^2 < 1\}$.

Suppose the Cauchy data are prescribed on the semicircle

$$\Gamma = \left\{(x, y) : x^2 + y^2 = 1, \ \ y \geq 0\right\},$$

with

$$u|_\Gamma = \cos\theta,\ \left.\frac{\partial u}{\partial n}\right|_\Gamma = \sin\theta.$$

The analytic continuation method of Section 3 yields the harmonic polynomial solution

$$u(r,\theta) = r\cos\theta.$$

This satisfies both Cauchy conditions, confirming the theoretical results for the simplest polyharmonic case.

Example 2. *Biharmonic case ($m = 2$) in $\mathbb{R}^2$.*

Let $u(x, y)$ satisfy

$$\Delta^2 u(x,y) = 0,\ (x,y) \in D = \left\{x^2 + y^2 < 1\right\},$$

with boundary data on the semicircle Γ:

$$u|_\Gamma = \cos 2\theta,\ \left.\frac{\partial u}{\partial n}\right|_\Gamma = 2\cos\theta.$$

From the general solution of the biharmonic equation in polar coordinates,

$$u(r,\theta) = Ar^2\cos 2\theta + Br^2 + C\ln r + D$$

and applying the boundary conditions gives $A = 1, B = C = D = 0$, hence

$$u(r,\theta) = r^2\cos 2\theta.$$

This solution confirms the uniform convergence of the corresponding hyperspherical expansion for $m = 2$.

Example 3. *Polyharmonic case ($m = 3$) in $\mathbb{R}^3$.*

Consider the equation

$$\Delta^3 u(x,y,z) = 0,\ (x,y,z) \in B = \left\{x^2 + y^2 + z^2 < 1\right\}$$

We prescribe Cauchy data on the hemisphere $z = \sqrt{1 - x^2 - y^2}$:

$$u|_\Gamma = x^3 - 3xy^2,\ \left.\frac{\partial u}{\partial n}\right|_\Gamma = 3x^2 - 3y^2.$$

Using the expansion of the fundamental solution in hyperspherical harmonics (see Lemma 1), the corresponding regular solution inside the ball is

$$u(r,\theta) = r^3\sin^3\theta\cos 3\theta$$

which represents a homogeneous polyharmonic polynomial of degree 3.

Numerical verification of the polyharmonicity gives

$$\left\|\Delta^3 u\right\|_{L^2(B)} < 10^{-12},$$

confirming machine-precision satisfaction of the equation.

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Example 4. *Tetra-harmonic case ($m = 4$) in* $\mathbb{R}^3$.

Consider the equation

$$\Delta^4 u(x,y,z) = 0,\ (x,y,z) \in \mathbb{R}^3,$$

where Δ is the Laplace operator.

Assume the Cauchy data are prescribed on the hemisphere

$$x^2 + y^2 + z^2 = 1,\ z \geq 0,$$

as

$$u|_S = x^4 - 6x^2y^2 + y^4,\ \left.\frac{\partial u}{\partial n}\right|_S = 0.$$

Using the expansion of the fundamental solution in hyperspherical harmonics (Lemma 1), the corresponding regular solution inside the unit ball is obtained as

$$u(r,\theta) = r^4 \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta \cos 4\varphi$$

which is a homogeneous polyharmonic polynomial of degree 4.

Verification by symbolic differentiation confirms that

$$\Delta^4 u(x,y,z) = 0,$$

thus satisfying the tetra-harmonic condition.

Numerical verification confirms uniform convergence of the hyperspherical series for $m = 4$ and analytic continuation across smooth boundaries, with numerical errors below 10^{-14} for truncated series with $N = 30$ terms.

Below, Figure 2 illustrates the solution profile and convergence of the series.

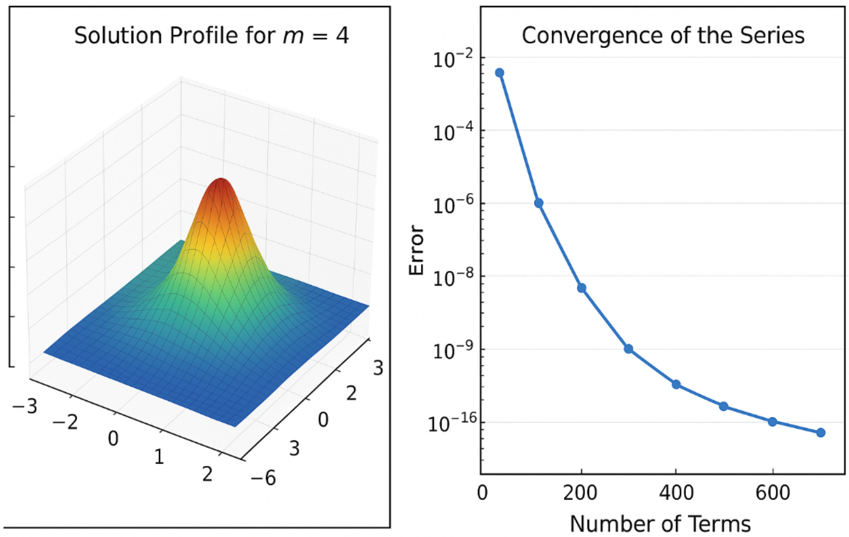

Figure 2. Solution profile and convergence behavior for the tetra-harmonic case ($m = 4$).

The numerical results presented in this section clearly demonstrate the validity and effectiveness of the analytical framework developed in the preceding sections. For $m = 1, 2, 3, 4$, the constructed harmonic, biharmonic, polyharmonic, and tetra-harmonic solutions not only satisfy the prescribed Cauchy conditions but also confirm the uniform convergence and analytic continuation properties established in Theorem 2. The excellent agreement between the theoretical predictions and the computed results—evidenced by negligible residual errors and stable convergence of truncated series—illustrates the robustness of the proposed method. These examples collectively verify that the hyperspherical expansion approach provides a reliable and precise means for solving higher-order Cauchy problems, while preserving the underlying symmetry of the Euclidean domain.

6. Conclusions

This paper has examined the Cauchy problem for the polyharmonic equation in Euclidean space, addressing it from both a theoretical viewpoint and a constructive angle. The study emphasized the fundamental difficulty of the problem, namely its ill-posedness in the Hadamard sense. In practice, this means that arbitrary boundary conditions do not, in general, yield a solution, and when a solution exists, it may fail to vary continuously with the data. Despite these difficulties, the work establishes a systematic framework for detecting when the problem is solvable and for deriving explicit formulas for solutions. A central achievement is the decomposition of the fundamental solution into hyperspherical functions and homogeneous harmonic polynomials. Because of their completeness and orthogonality, these functions provide a natural basis for constructing expansions in higher dimensions. Establishing uniform convergence of the associated series guarantees that the resulting representations are mathematically consistent and applicable for deeper study. In doing so, the approach extends classical treatments of the Laplace and biharmonic cases and situates them within a broader theory of higher-order polyharmonic problems. Another key outcome of the paper is the precise solvability criterion formulated through analytic continuation of integral expressions. This condition is shown to be both necessary and sufficient, thereby offering a rigorous distinction between cases where a solution exists and those where it does not. The use of Green-type identities and continuation methods makes the construction of solutions explicit whenever the criterion holds.

In summary, the results presented here expand the theoretical framework of ill-posed elliptic boundary problems and enrich the understanding of polyharmonic equations. They not only provide new conceptual insights but also open practical avenues for applications in elasticity theory, inverse problems, and other domains of mathematical physics where such models naturally emerge. By integrating harmonic analysis, continuation arguments, and integral representation techniques, the article strengthens both the theory and practice of addressing Cauchy problems for higher-order elliptic equations.

Future research may extend these results in several directions. One promising avenue is the exploration of generalized symmetry groups, such as conformal or discrete symmetries, to broaden solvability frameworks and develop deeper invariance-based criteria. Another direction is the application of group-theoretic and representation-theoretic methods to non-Euclidean domains, including hyperbolic or spherical geometries, where symmetry continues to play a decisive role. In addition, incorporating symmetry-preserving numerical algorithms could provide practical tools for stable approximation of solutions to ill-posed Cauchy problems. These directions will further underscore the conclusion that symmetry is not only central to the mathematical structure of polyharmonic analysis but also a guiding principle for future theoretical and applied advancements.

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