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Boundary Value Problems

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On a non-self-adjoint Sturm–Liouville problem: spectral and inverse analysis

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Abstract

In this paper, we study a Sturm–Liouville boundary value problem with a complex-valued potential represented by an absolutely convergent Fourier series. We construct fundamental solutions, define the characteristic function, and analyze the discrete spectrum. By extending the potential to the half-line, we introduce Jost-type solutions and derive the associated spectral coefficient. Furthermore, we investigate the Green function and resolvent operator, proving compactness and discreteness of the spectrum.

Keywords: Discontinuous equation; Spectral singularities; Non-self-adjoint problem; Inverse spectral problem

1 Introduction

The spectral theory of Sturm–Liouville operators constitutes a fundamental area in mathematical physics, differential equations, and operator theory, with wide-ranging applications in quantum mechanics, wave propagation, and inverse spectral problems.

In this paper, a central object of study is the differential equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \tag{1.1}$$

on the space $L_2(0,a)$, where

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty, \tag{1.2}$$

with boundary conditions

$$y(0) = 0, \qquad y'(a) + hy(a) = 0. \tag{1.3}$$

defined on a finite interval together with appropriate boundary conditions.

Recently, non-self-adjoint Sturm–Liouville problems with complex-valued potentials have attracted significant attention, especially when the potential admits a structured representation. Among these, potentials expressed via absolutely convergent Fourier series form an important result. The significance of this result lies in the fact that it provides an explicit representation of solutions in terms of exponential functions and Fourier coefficients of the potential. Classical results due to Titchmarsh [23], Levitan and Sargsjan [19], Marchenko [20], and Naimark [21] provide a comprehensive framework for the study of self-adjoint second-order differential operators, including the structure of eigenvalues, eigenfunction expansions, and completeness properties.

In the non-self-adjoint setting, the theory becomes substantially more intricate due to the possible lack of orthogonality and the appearance of complex spectra. A significant contribution in this direction was made by Gasymov [16], who investigated a class of Sturm–Liouville operators with complex-valued Fourier-type potentials and obtained explicit representations of solutions. These results opened the way for the study of operators with structured potentials, where harmonic analysis techniques can be effectively applied.

The novelty of this work lies in extending Gasymov-type potentials to a finite interval setting with boundary conditions, where the spectrum is discrete and the inverse problem is solved via boundary spectral data.

Further developments in the spectral analysis of non-self-adjoint operators, including basis properties and spectral asymptotics, were obtained by Veliev and Shkalikov [26, 27] and Aktosun [1], as well as by Djakov and Mityagin [9]. The spectral structure of related periodic and Hill-type operators has also been investigated in detail by Tkachenko [24] and Korotyaev [18], revealing important features of operators with complex and periodic potentials.

Inverse spectral problems, which aim to reconstruct the potential from spectral data, constitute another fundamental direction. Classical uniqueness results were established by Borg [5], while modern approaches and constructive methods have been developed by Yurko [28] and Freiling and Yurko [14]. Extensions to operators with singular or distributional potentials were studied by Savchuk and Shkalikov [22] and Hryniv and Mykytyuk [17].

In addition, Sturm–Liouville problems with discontinuities, non-smooth coefficients, and spectral parameters in the boundary conditions have been investigated by Efendiev [3, 4, 10–13], where spectral properties and inverse problems were analyzed in non-regular settings and [2, 6–8, 15, 25]. These works highlight the influence of structural irregularities on the spectral characteristics of differential operators.

In this paper, we investigate a Sturm–Liouville boundary value problem on a finite interval with a complex-valued potential represented by an absolutely convergent Fourier series. While the existence of solutions for such potentials follows from classical theory, our approach provides a unified and constructive treatment of the associated spectral characteristics.

The paper is organized as follows. In Sect. 2, we establish the existence and properties of fundamental solutions. Section 3 is devoted to the spectral analysis, including the construction of the characteristic function, Jost-type solutions, and the spectral coefficient. The Green function and resolvent operator are also derived in this section. In Sect. 4, we address the inverse problem and present a constructive method for recovering the potential from spectral data.

2 Existence of fundamental solutions

In this section, we establish the existence and basic properties of fundamental solutions for the Sturm–Liouville equation under consideration. The construction of such solutions plays a central role in the spectral analysis of the problem, as they form a basis for representing all solutions of the differential equation and are essential in defining the characteristic function and spectral data.

We assume that the potential $q(x)$ admits an absolutely convergent Fourier series representation. This assumption ensures that $q \in L_1(0,a)$, which allows us to apply standard results from the theory of linear differential equations with integrable coefficients. In particular, the equation can be rewritten as an equivalent first-order system with summable coefficients, guaranteeing existence and uniqueness of solutions for any prescribed initial data.

Under these conditions, we now state the existence theorem for the fundamental system of solutions.

Theorem 1 (Existence of Fundamental Solutions) *Consider the Sturm–Liouville equation*

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \qquad x \in (0,a). \tag{2.1}$$

Assume that the potential $q(x)$ admits the Fourier representation

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty. \tag{2.2}$$

Then, for every fixed $\lambda \in \mathbb{C}$, there exist unique solutions $\phi(x,\lambda)$ and $\theta(x,\lambda)$ satisfying the initial conditions

$$\varphi(0,\lambda) = 0, \qquad \varphi'(0,\lambda) = 1, \tag{2.3}$$

$$\theta(0,\lambda) = 1, \qquad \theta'(0,\lambda) = 0, \tag{2.4}$$

respectively. Moreover, these solutions exist on the entire interval $[0,a]$ and are absolutely continuous together with their first derivatives.

Proof Since

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty,$$

the series converges absolutely, hence $q \in L_1(0,a)$. Therefore, the differential equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x)$$

can be rewritten as

$$y''(x) = \big(q(x) - \lambda^2\big)y(x).$$

Introduce

$$Y(x) = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}.$$

Then the equation becomes the first-order system

$$Y'(x) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda^2 & 0 \end{pmatrix} Y(x).$$

Since the coefficient matrix belongs to $L_1(0,a)$, the standard existence and uniqueness theorem for linear systems implies that, for any initial vector $Y(0)$, the Cauchy problem has a unique absolutely continuous solution on $[0,a]$.

Choosing

$$Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

we obtain the solutions $\phi(x,\lambda)$ and $\theta(x,\lambda)$, respectively. □

Before proceeding to the construction of Jost-type solutions, we recall a fundamental result due to Gasymov, which plays a crucial role in the spectral analysis of Sturm–Liouville operators with Fourier-type potentials.

The significance of this result lies in the fact that it provides an explicit representation of solutions in terms of exponential functions and Fourier coefficients of the potential. In contrast to general potentials, the special structure of $q(x)$ as an absolutely convergent Fourier series allows one to construct solutions in a form resembling plane waves, modified by an infinite series reflecting the interaction with the potential.

This representation can be viewed as a non-self-adjoint analogue of scattering-type solutions. The function $f(x,\lambda)$ behaves like the free solution $e^{i\lambda x}$ multiplied by a correction term.

The convergence of the associated series ensures that the solution is well-defined and analytically tractable.

We now state Gasymov's theorem for ease of reading.

Theorem 2 (*Gasymov's theorem*) *Let $q(x)$ be of the form* (1.2). *Then the equation* (1.1) *has a solution of the form*

$$f(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2\lambda} \sum_{\alpha=n}^{\infty} \nu_{n\alpha} e^{i\alpha x}\right), \tag{2.5}$$

where

$$\alpha(\alpha - n)\nu_{n\alpha} + \sum_{s=n}^{\alpha-1} q_{\alpha-s}\nu_{ns} = 0, \quad 1 \le n < \alpha, \tag{7}$$

$$\alpha \sum_{n=1}^{\alpha} \nu_{n\alpha} + q_\alpha = 0. \tag{8}$$

and series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha(\alpha - n)|\nu_{n\alpha}|, \tag{2.6}$$

$$\sum_{n=1}^{\infty} n|\nu_{nn}| \tag{2.7}$$

converge.

Proof The proof of the theorem can be found in Gasymov [7]. □

Remark 1 Gasymov's original construction was formulated for the entire axis problem and relies on the asymptotic condition

$$f(x,\lambda) \sim e^{i\lambda x}, \qquad x \to \infty.$$

In the present work, the problem is posed on the finite interval $(0,a)$. Nevertheless, the representation obtained in Gasymov's theorem remains applicable after extending the potential by zero outside $(0,a)$. Indeed, defining

$$\widetilde{q}(x) = \begin{cases} q(x), & x \in (0,a], \\ 0, & x > a, \end{cases}$$

one recovers a half-line problem for which the same Jost-type solution exists. On the interval $[0,a]$, the solution preserves the representation stated above, while for $x > a$ it coincides with the free wave $e^{i\lambda x}$. Thus, Gasymov's construction adapts naturally to the present finite-interval setting and provides the analytic framework used in the subsequent spectral analysis.

Remark 2 The functions $f(x,\lambda)$ and $f(x,-\lambda)$ are linearly independent, and their Wronskian equals $2i\lambda$.

Define

$$f_n(x) = \lim_{\lambda \to -n/2} (n - 2\lambda) f(x,-\lambda) = \sum_{\alpha=n}^{\infty} \nu_{n\alpha} e^{i\alpha x} e^{-inx/2}.$$

If $\nu_{nn} = 0$, then $\nu_{n\alpha} = 0$ for $\alpha > n$, hence $f_n(x) \equiv 0$.

Remark 3 From Remark 1 it follows that the Wronskian of $f(x,n/2)$ and $f_n(x)$ is zero; hence they are linearly dependent. Therefore,

$$f_n(x) = s_n f(x,n/2), \tag{2.8}$$

and comparison of formulas shows that $s_n = \nu_{nn}$.

In order to further analyze the spectral properties of the problem and establish a connection with scattering-type methods, it is convenient to extend the potential beyond the finite interval $(0,a)$.

Specifically, we consider a zero extension of the potential to the half-line $[0,\infty)$, which allows us to embed the original boundary value problem into a larger framework where asymptotic analysis becomes applicable. This extension preserves the integrability of the potential and enables the use of techniques commonly employed in the study of non-self-adjoint operators and inverse problems.

Within this extended setting, one can construct special solutions that exhibit prescribed asymptotic behavior at infinity. These solutions, known as Jost-type solutions, play a central role in spectral and scattering theory, as they provide a natural basis for describing the interaction between the potential and propagating waves.

In the present case, due to the compact support of the extended potential, the asymptotic behavior simplifies significantly: beyond the interval $[0,a]$, the equation reduces to the free equation, and the Jost-type solution coincides exactly with a plane wave. This feature allows for an explicit and constructive formulation of the solution.

We now establish the existence and uniqueness of such a Jost-type solution.

Theorem 3 (Existence of a Jost-Type Solution After Zero Extension) *Consider the Sturm–Liouville problem* (1.1)-(1.3).

Extend the potential to the half-line $[0,\infty)$ *by*

$$\widetilde{q}(x)=\begin{cases} q(x), & x\in(0,a],\\ 0, & x>a.\end{cases}$$

Then $\widetilde{q}\in L_1(0,\infty)$, *and for each* $\lambda\in\mathbb{C}$ *with* $\operatorname{Im}\lambda\geq 0$ *and* $\lambda\neq 0$, *there exists a unique solution* $f(x,\lambda)$ *of*

$$-y''(x)+\widetilde{q}(x)y(x)=\lambda^2 y(x),\qquad x\in(0,\infty),\tag{2.9}$$

satisfying

$$f(x,\lambda)\sim e^{i\lambda x},\qquad x\to\infty.\tag{2.10}$$

Moreover, for $x\geq a$,

$$f(x,\lambda)=e^{i\lambda x}.\tag{2.11}$$

This solution is called the Jost-type solution.

Proof Since $\widetilde{q}(x)=0$ for $x>a$, the equation reduces there to

$$-y''(x)=\lambda^2 y(x).$$

Hence, on (a,∞), the solution satisfying the outgoing condition at infinity is exactly

$$f(x,\lambda)=e^{i\lambda x}.$$

Therefore,

$$f(a,\lambda)=e^{i\lambda a},\qquad f'(a,\lambda)=i\lambda e^{i\lambda a}.$$

Now solve the differential equation backward on $[0,a]$ with these terminal conditions. Since $\widetilde{q} = q \in L_1(0,a)$, the standard existence and uniqueness theorem yields a unique absolutely continuous solution on $[0,a]$. Combining this with the free solution on $[a,\infty)$ gives a unique solution on $[0,\infty)$ satisfying the required asymptotic condition. □

Remark 4 Since the potential vanishes identically on (a,∞), the Jost-type solution is not only asymptotic to the free wave but actually coincides with it:

$$f(x,\lambda) = e^{i\lambda x}, \qquad x > a.$$

Therefore, on the interval $[0,a]$, the function $f(x,\lambda)$ is determined by solving the equation backward from the terminal data

$$f(a,\lambda) = e^{i\lambda a}, \qquad f'(a,\lambda) = i\lambda e^{i\lambda a}.$$

3 Spectrum

Every solution of the equation can be represented in the form

$$y(x,\lambda) = C_1\varphi(x,\lambda) + C_2\theta(x,\lambda). \tag{3.1}$$

Imposing the boundary condition $y(0) = 0$, we obtain $C_2 = 0$. Therefore, every eigenfunction is proportional to $\varphi(x,\lambda)$.

Substituting the solution $\varphi(x,\lambda)$ into the boundary condition at $x = a$, we obtain

$$\varphi'(a,\lambda) + h\varphi(a,\lambda) = 0. \tag{3.2}$$

This motivates the definition of the characteristic function

$$\Delta(\lambda) := \varphi'(a,\lambda) + h\varphi(a,\lambda). \tag{3.3}$$

Eigenvalues correspond to the condition

$$\Delta(\lambda) = 0, \tag{3.4}$$

with operator eigenvalues given by $\mu = \lambda^2$.

Thus, the spectrum of L is

$$\sigma(L) = \{\lambda^2 : \Delta(\lambda) = 0\}. \tag{3.5}$$

Since $\Delta(\lambda)$ is an entire function, its zeros form a discrete set without finite accumulation points. Therefore, the spectrum is discrete.

Theorem 4 *The spectrum $\sigma(L)$ is purely discrete*:

$$\sigma(L) = \left\{\lambda^2 \in \mathbb{C} : \Delta(\lambda) = 0\right\}, \tag{3.6}$$

where

$$\Delta(\lambda) := \varphi'(a,\lambda) + h\,\varphi(a,\lambda) \tag{3.7}$$

is the characteristic function, and $\varphi(x,\lambda)$ is the regular solution satisfying

$$\varphi(0,\lambda) = 0, \qquad \varphi'(0,\lambda) = 1. \tag{3.8}$$

Since $q \in L^1(0,a)$, the solution $\varphi(x,\lambda)$ is absolutely continuous and analytic in λ for each fixed $x \in [0,a]$. Therefore $\Delta(\lambda)$ is an entire function of λ. Being nontrivial, its zeros form a discrete set in $\mathbb{C}$ with no finite accumulation point, and all eigenvalues have finite algebraic multiplicity.

3.1 Jost-type solution

We extend the potential by zero to the half-line:

$$q_e(x) = \begin{cases} q(x), & 0 < x \leq a, \\ 0, & x > a. \end{cases} \tag{3.9}$$

Then $q_e \in L_1(0,\infty)$, and for $x > a$ the equation becomes

$$-y''(x) = \lambda^2 y(x). \tag{3.10}$$

Therefore, the outgoing solution on (a,∞) is

$$f(x,\lambda) = e^{i\lambda x}. \tag{3.11}$$

Solving the equation backward on $[0,a]$ with these terminal conditions yields a unique solution on $[0,a]$. Thus, for every λ with $\operatorname{Im}\lambda \geq 0$, there exists a unique Jost-type solution satisfying

$$f(x,\lambda) \sim e^{i\lambda x}, \qquad x \to \infty, \tag{3.12}$$

and in fact

$$f(x,\lambda) = e^{i\lambda x}, \qquad x > a, \tag{3.13}$$

because the extended potential vanishes identically for $x > a$.

In the same spirit as the paper, one may also introduce the second free solution

$$\widetilde{f}(x,\lambda) = e^{-i\lambda x}, \qquad x > a, \tag{3.14}$$

and extend it backward to $[0,a]$. The pair $f(x,\lambda)$ and $\widetilde{f}(x,\lambda)$ forms a Jost-type basis. Their Wronskian is constant and can be computed for $x > a$:

$$W[f,\widetilde{f}] = f(x,\lambda)\widetilde{f}'(x,\lambda) - f'(x,\lambda)\widetilde{f}(x,\lambda) = -2i\lambda. \tag{3.15}$$

Hence, for $\lambda \neq 0$, these solutions are linearly independent.

Since $\{\varphi(x,\lambda),\theta(x,\lambda)\}$ and $\{f(x,\lambda),f(x,-\lambda)\}$ are both fundamental systems, the regular solution can be represented in the Jost basis:

$$\varphi(x,\lambda) = A(\lambda)f(x,\lambda) + B(\lambda)f(x,-\lambda). \tag{3.16}$$

Applying the boundary form

$$U_a(y) := y'(a) + hy(a), \tag{3.17}$$

we obtain

$$0 = U_a(\varphi) = A(\lambda)U_a(f(x,\lambda)) + B(\lambda)U_a(f(x,-\lambda)) \tag{3.18}$$

for eigenvalues. Define

$$\Phi^+(\lambda) := U_a(f(x,\lambda)) = f'(a,\lambda) + hf(a,\lambda), \tag{3.19}$$

$$\Phi^-(\lambda) := U_a(f(x,-\lambda)) = f'(a,-\lambda) + hf(a,-\lambda). \tag{3.20}$$

Thus, the coefficients satisfy

$$A(\lambda)\Phi^+(\lambda) + B(\lambda)\Phi^-(\lambda) = 0 \tag{3.21}$$

at eigenvalues.

This leads to the spectral coefficient

$$S(\lambda) := \frac{A(\lambda)}{B(\lambda)} = -\frac{\Phi^-(\lambda)}{\Phi^+(\lambda)} = -\frac{f'(a,-\lambda) + hf(a,-\lambda)}{f'(a,\lambda) + hf(a,\lambda)}. \tag{3.22}$$

This formula describes the relation between incoming and outgoing components of the regular solution.

4 Inverse problem

In this section, we formulate and solve the inverse spectral problem for the Sturm–Liouville operator introduced in Sect. 1. The goal is to reconstruct the potential $q(x)$ from spectral data associated with the boundary value problem.

Definition. The spectral data of the non-self-adjoint Bessel-type Sturm–Liouville operator are defined as

$$\mathcal{S} = \left\{S(\lambda),\ \pm\frac{n}{2}, n \in N\right\}$$

where $S(\lambda)$ is the spectral coefficient defines in (3.22), $\pm\frac{n}{2}, n \in N$ denotes the corresponding resonance.

Inverse Problem Given the spectral data

$$S(\lambda) := \frac{A(\lambda)}{B(\lambda)} = -\frac{f'(a,-\lambda) + hf(a,-\lambda)}{f'(a,\lambda) + hf^{(}a,\lambda)}\ \textit{and}\ \pm\frac{n}{2}, n \in N$$

reconstructs the potential $q(x)$.

We begin by showing that the diagonal coefficients V_{nn}, $n \in \mathbb{N}$, can be uniquely recovered from the spectral data $S(\lambda)$.

Taking the limit as $\lambda \to \frac{n}{2}$, we obtain

$$\lim_{\lambda \to \frac{n}{2}} (n-2\lambda)S(\lambda) = \lim_{\lambda \to \frac{n}{2}} (n-2\lambda) - \frac{f'(a,-\lambda) + hf(a,-\lambda)}{f'(a,\lambda) + hf(a,\lambda)}$$
$$= \frac{\lim_{\lambda \to \frac{n}{2}} (n-2\lambda) f'(a,-\lambda) + h \lim_{\lambda \to \frac{n}{2}} (n-2\lambda) f(a,-\lambda)}{f'\left(a,\frac{n}{2}\right) + hf\left(a,\frac{n}{2}\right)} = S_n.$$

Since

$$S_n = v_{nn},$$

the coefficients v_{nn}, $n \in \mathbb{N}$, are uniquely determined by the spectral data.

Once the diagonal coefficients V_{nn} are known, the recursive relations

$$\alpha(\alpha - n)v_{n\alpha} = \sum_{s=n}^{\alpha-1} q_{\alpha-s} v_{ns}, \qquad \alpha > n,$$

allow one to determine all coefficients $v_{n\alpha}$ successively.

Assume that $q_1, \dots, q_{m-1}$ and all coefficients $V_{n\beta}$ with $\beta < m$ are already determined. Then the above relation uniquely determines V_{nm} for $n < m$. Since V_{mm} is known from the spectral data, the identity

$$q_m = m \sum_{n=1}^{m} V_{nm}$$

yields the coefficient q_m.

Proceeding inductively, all Fourier coefficients q_m, $m \geq 1$, are uniquely recovered. Consequently, the potential is reconstructed in the form

$$q(x) = \sum_{m=1}^{\infty} q_m e^{imx}.$$

Theorem 5 *The spectral data $S(\lambda)$ uniquely and constructively determine the potential $q(x)$.*

4.1 Conclusion

Thus, the boundary value problem

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \qquad y(0) = 0, \qquad y'(a) + hy(a) = 0, \tag{4.1}$$

is solved in the following sense:

1. The fundamental solutions $\varphi(x,\lambda)$ and $\theta(x,\lambda)$ exist uniquely on $[0,a]$.
2. The eigenvalues are determined by the zeros of the characteristic function

$$\Delta(\lambda) = \varphi'(a,\lambda) + h\varphi(a,\lambda).$$

3. After zero extension of the potential, one constructs a Jost-type solution satisfying

$$f(x,\lambda) \sim e^{i\lambda x}, \qquad x \to \infty.$$

4. The regular solution admits a representation in the Jost basis, leading to the spectral coefficient

$$-\frac{f'(a,-\lambda) + hf(a,-\lambda)}{f'(a,\lambda) + hf(a,\lambda)}$$

5. The Green function and resolvent are investigated.
6. The resolvent is compact, hence the spectrum is purely discrete.

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Declarations

Competing interests
The authors declare no competing interests.

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