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Least energy solutions for Choquard equations involving vanishing potentials and exponential growth

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Abstract

In this paper, we consider the existence of solutions for Choquard equation of the form

$$-\Delta u + V(|x|)u = [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u), \quad x \in \mathbb{R}^2,$$

where the nonlinear term f has exponential growth, the radial potentials V, $Q : \mathbb{R}^+ \to \mathbb{R}$ are unbounded, singular at the origin or decaying to zero. By combining the variational methods, Trudinger-Moser inequality and some new approaches to estimate precisely the minimax level of the energy functional, we prove the existence of a nontrivial solution for the above problem under some weaker assumptions. Our study extends and improves the results of [Albuquerque-Ferreira-Severo, Milan J. Math. 89 (2021)] and [Alves-Shen, J. Differential Equations, 344 (2023)].

Keywords Choquard equations · Critical exponential growth · Trudinger-Moser inequality

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1 Introduction and main results

In the present paper, we consider the existence of solutions for nonlinear Choquard equations of the form

$$- \Delta u + V(|x|)u = [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u), \quad x \in \mathbb{R}^2, \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $F(t) := \int_0^t f(s)ds$, the radial potentials $V, Q : \mathbb{R}^+ \to \mathbb{R}$ are unbounded, singular at the origin or decaying to zero at infinity. The Riesz potential I_α with order $\alpha \in (0, 2)$ is defined for $x \in \mathbb{R}^2 \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{2-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma(\frac{2-\alpha}{2})}{2^\alpha \pi \Gamma(\frac{\alpha}{2})}.$$

We recall here that Γ is the classical Gamma function and $*$ denotes the convolution on $\mathbb{R}^2$. Explicitly, we introduce the following assumptions on V and f:

(V1) $V \in C((0, +\infty), (0, +\infty))$, and there exist $a_0, a > -2$ such that

$$\limsup_{r \to 0^+} \frac{V(r)}{r^{a_0}} < +\infty, \quad \liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0;$$

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, and

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > 0;$$

or

(F1′) $f \in C(\mathbb{R}, \mathbb{R})$, and there exists $\beta_0 > 0$ such that

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > \beta_0$$

and

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \text{for all } \beta < \beta_0;$$

(F2) $\lim_{t \to 0} \left| \frac{f(t)}{t^{\alpha/2}} \right| = l \in [0, +\infty)$;

(F3) there exists $\mu > 1$ such that

$$f(t)t \geq \mu F(t) > 0, \quad \forall t \in \mathbb{R} \setminus \{0\};$$

(F4) there exist $M_0 > 0$ and $t_0 > 0$ such that

$$F(t) \leq M_0|f(t)|, \quad \forall |t| \geq t_0.$$

To facilitate the subsequent analysis, we introduce the following notations. Let

$$E := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : u \text{ is radial}, |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(|x|)u^2 dx < +\infty \right\}$$

and define

$$(u, v) := \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(|x|)uv) \, dx, \quad \forall u, v \in E; \quad \|u\|^2 := \sqrt{(u, u)} \quad \forall u \in E.$$

Then E is a Hilbert space with the above inner product $(\cdot, \cdot)$ (see [1, Proposition 2.1]). Furthermore, for $s \in [1, +\infty)$, we define

$$L^s(\mathbb{R}^2, Q) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is mensurable, } \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^s \mathrm{d}x < \infty \right\}.$$

It is widely known that the term $|x|^{\alpha-2} * (Q(x)F(u))$ can be regarded as the convolution between the Riesz potential $|x|^{\alpha-2}$ and $Q(x)F(u)$. Thus, problem (1.1) is closely related to the Choquard equation arising from the study of Bose-Einstein condensation and can be exploited to describe the finite-range many-body interactions between particles. For $N \geq 3$, the Choquard equation under the convolution of the Riesz potential is simply of the form

$$- \Delta u + u = \left(|x|^{\alpha-N} * |u|^p \right) u^{p-2} u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

In the relevant physical case in which $N = 3$, $\alpha = 1$ and $p = 2$, equation (1.2) turns into the Choquard-Pekar equation, which was used by Pekar [2] to describe a polaron at rest in the quantum field theory. It was also investigated by Choquard to characterize an electron trapped in its own hole as an approximation to the Hartree-Fock theory for a one component plasma [3]. Subsequently, Lieb [4] and Lions [5] obtained the existence and uniqueness of positive solutions to (1.2) by variational methods. It should be pointed out that equation (1.2) was also proposed by Moroz-Penrose-Tod in [6] as a model for self-gravitating particles where it can be viewed as the classical Schrödinger-Newton equation, see e.g. [7–9].

Problem (1.2) has attracted a lot of interest in recent years and has been extensively investigated in the literature, such as, the existence and multiplicity of nontrivial solutions in [10–12], the sign-changing solutions in [13, 14] and the semiclassical solutions in [15, 16]. We also refer to reference [17] for a broad survey of the Choquard equations. We emphasize that all the results mentioned above require that the exponent $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$, which is deduced from the Hardy-Littlewood-Sobolev inequality (see Lemma 2.1) and the Sobolev embedding theorem.

Notably, the case $N = 2$ is very special, as the corresponding Sobolev embedding yields $H^1(\mathbb{R}^2) \subset L^s(\mathbb{R}^2)$ for all $s \in [2, +\infty)$, but $H^1(\mathbb{R}^2) \nsubseteq L^\infty(\mathbb{R}^2)$. In dimension $N = 2$, in order to address problems with exponential growth, one of the most important tools is the Trudinger-Moser inequality, which can be seen as a substitute of the Sobolev inequality. The first version of the Trundiger-Moser inequality in $\mathbb{R}^2$ was established by Cao in [18], see also [19, 20] and reads as follow:

i) if $\beta > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1 \right) \mathrm{d}x < \infty;$$

ii) if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2 \leq M < \infty$, and $\beta < 4\pi$, then there exists a constant $\mathcal{C}(M, \beta)$, which depends only on M and β, such that

$$\int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1 \right) \mathrm{d}x \leq \mathcal{C}(M, \beta).$$

Based on Trundiger-Moser inequality, we say that f has subcritical exponential growth on $\mathbb{R}^2$ at infinity if (F1) holds, and f has critical exponential growth on $\mathbb{R}^2$ at infinity if (F1′) holds. Now let us recall some related works for (1.1) with subcritical or critical exponential growth.

When $Q(x) \equiv 1$, (1.1) reduces to the following Choquard equation

$$- \Delta u + V(x)u = (I_\alpha * F(u)) f(u), \quad x \in \mathbb{R}^2. \tag{1.3}$$

Consider the case $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^2$. In [21], Alves-Cassani-Tarsi-Yang studied (1.3) with the critical exponent $\beta_0 = 4\pi$ under (F3) and the following assumptions on f:

(f_1) $f(t) = 0$ for $t \leq 0$, $0 \leq f(t) \leq C e^{4\pi t^2}$, $\forall\, t > 0$;
(f_2) there exists $p > \frac{\alpha}{2}$ and $C_p > 0$ such that $f(t) \sim C_p t^p$, as $t \to 0$;
(f_3) there exist $t_1 > 0$, $M_1 > 0$ and $q \in (0, 1]$ such that $0 < t^q F(t) \leq M_1 f(t)$, $\forall\, t \geq t_1$;
(f_4) $\displaystyle\lim_{t \to \infty} \frac{t f(t) F(t)}{e^{8\pi t^2}} \geq k$ with $k > \displaystyle\inf_{\rho > 0} \frac{\alpha(1+\alpha)(2+\alpha)^2}{16\pi^2 \rho^{2+\alpha}} e^{\frac{2+\alpha}{4} V_\rho \rho^2}$, where $V_\rho :=$ $\displaystyle\sup_{|x| \leq \rho} V(x)$.

They obtained nontrivial solutions for (1.3) by using the mountain pass lemma and showing that the mountain pass level shall be less than $\frac{2+\alpha}{8}$ provided that the potential V is periodic or satisfies the Rabinowitz type condition introduced in [22]. We point out that the threshold of the mountain pass level can be directly deduced from the condition (f_4). By considering a sequence of measures which have uniformly bounded total variation and using the Radon-Nicodym theorem, they [21] showed that the weak limit of a Palais-Smale sequence for the energy functional associated with (1.3) is a solution, moreover, the weak limit is proved to be nonzero by using the condition (f_4) which constitutes the core of the proof. For the case that V change sign, Qin-Tang [23] developed a direct approach to deal with the equation (1.3) with both critical exponential growth and strongly indefinite features when 0 lies in a gap of the spectrum of the operator $-\Delta + V$. They proved the existence of nontrivial solutions for equation (1.3) under (F1$'$), (F2) with $l = 0$, (F3), (F4) and the following assumption:

(f_5) $\displaystyle\liminf_{t \to \infty} \frac{f(t)}{e^{\beta_0 t^2}} > \frac{\sqrt{\alpha(1+\alpha)}(2+\alpha)}{\sqrt{2\pi A_\alpha} \rho^{1+\alpha/2}} e^{4(2+\alpha)\pi(1+\rho)^2 \mathcal{B}_0^2/(2+\rho)}$ where $\rho > 0$ satisfies $2(2+\alpha)\pi\rho^2 \mathcal{B}_0^2 < 1$ and $\mathcal{B}_0 > 0$ is an embedding constant.

(f_5) in [23] plays the same role like (f_4) in [21]and enables taking advantage of the Moser functions to pull down the critical value to a particular threshold value $\frac{(2+\alpha)\pi}{2\beta_0}$ and then showing that the weak limit of a Palais-Smale sequence is nonvanishing. They also established the existence of ground states and geometrically distinct solutions for the equation (1.3) when the nonlinearity has subcritical exponential growth.

Compare with equation (1.3), the weight function Q will cause difficulties in estimating the threshold value of equation (1.1), and thus more efforts are required when studying the equation (1.1). Albuquerque-Ferreira-Severo[1] established a new version of the Trudinger-Moser inequality (see [1, Proposition 2.9]), which will plays an important role in their arguments. Based on the weighted Trudinger-Moser inequality, Albuquerque-Ferreira-Severo [1] proved that (1.1) has a nontrivial weak solution provided that V, Q and f satisfy (V1), (F1$'$), (f_3) and the following assumptions:

(Q1$'$) $Q \in \mathcal{C}((0, +\infty), (0, +\infty))$, and there exist $b_0 > -\frac{2+\alpha}{2}$ and $b < \frac{a(2+\alpha)}{4}$ such that

$$\limsup_{r \to 0^+} \frac{Q(r)}{r^{b_0}} < +\infty, \quad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < +\infty;$$

(Q2$'$) $\displaystyle\liminf_{r \to 0^+} \frac{Q(r)}{r^{b_0}} > 0$;

(f_6) $f \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and $\displaystyle\lim_{t \to 0^+} \frac{f(t)}{t^{\alpha/2}} = 0$;

(f_7) there exists $\theta > 1$ such that $\theta F(t) \le f(t)t$, $\forall t \ge 0$;

(f_8) $\kappa_* := \liminf\limits_{t \to +\infty} \dfrac{F(t)}{e^{\beta_0 t^2}} > 0.$

Later Shen-Rădulescu-Yang [24] studied the equation (1.1), where the potential V and the weight Q satisfy the following hypotheses:

(K) $V, Q \in \mathcal{C}(\mathbb{R}^2)$ and there exist some positive constants γ, τ, a, A and b such that

$$\frac{a}{1 + |x|^\gamma} \le V(x) \le A \text{ and } 0 < Q(x) \le \frac{b}{1 + |x|^\tau},$$

where $V(x), Q(x) \sim |x|^{-\tau}$ as $|x| \to +\infty$ and (γ, τ) satisfies one of the following assumptions:

(i) $0 < \gamma < 2$ and $(2 + \alpha)\gamma/4 \le \tau < +\infty$, or $0 < \gamma \le 4\tau/(2 + \alpha) < 2$;
(ii) $\gamma = 2$ and $(2 + \alpha)/2 \le \tau < +\infty$;
(iii) $\gamma > 2$ and $(2 + \alpha)/2 \le \tau < +\infty$;

and f satisfies (F1$'$), (f_3), (f_8) as well as

(f_9) $f \in \mathcal{C}^1(\mathbb{R})$, $f(t) \equiv 0$ for all $t \le 0$ and $\lim\limits_{t \to 0^+} \dfrac{f(t)}{t^{\alpha/2}} = 0$;

(f_{10}) there exists a constant $\delta \in [0, 1)$ such that $\dfrac{F(t)f'(t)}{f^2(t)} \ge \delta$, $\forall t > 0$.

They investigated the existence of nontrivial solutions of mountain-pass type for (1.1). Furthermore, they also proved that the nontrivial solution is a bound state, namely a solution belonging to $H^1(\mathbb{R}^2)$, for some particular (γ, τ).

We emphasize that hypothesis (f_8) is usually used to estimate the minimax level of the energy functional associated to the critical exponential growth problems. In [1, 24], hypothesis (f_8) is crucial to overcome the obstacles caused by the critical exponential term. Indeed, by (f_8), mountain-pass level c can be controlled by a fine threshold $\frac{\pi(2 + \alpha + 2b_0)}{2\beta_0}$ ($b_0 = 0$ in [24]) under which the compactness can be restored for the critical case (see [1, Proposition 6.2] and [24, Lemma 3.3]).

Inspired by the works mentioned above, two natural questions arise:

(Q1) Can we establish the existence of nontrival solution of (1.1) by using new assumptions on the weight function Q that are different from (Q1$'$) and (Q2$'$)?

(Q2) As we can see (f_8) is an essential technical condition for the critical exponential growth problems. When studying (1.1) in critical exponential case, can we weaken (f_8) to more general conditions?

The main purpose of this article is to address the above questions. Based on the above observations and inspired by the work [25], we shall further study the existence of nontrivial solutions for equation (1.1) under subcritical and critical exponential growth. Besides (V1), (F1), (F1$'$) and (F2)-(F4), we introduce the following assumptions:

(Q1) $Q \in \mathcal{C}((0, +\infty), (0, +\infty))$, and there exists $b_0 > -\frac{2+\alpha}{2}$ such that

$$\limsup_{r \to 0^+} \frac{Q(r)}{r^{b_0}} < +\infty, \quad \limsup_{r \to +\infty} \frac{Q(r)^{\frac{4}{2+\alpha}}}{V(r)} = 0;$$

(Q2) there holds

$$\liminf_{r \to 0^+} \frac{\int_0^r s Q(s) ds}{r^{2+b_0}} =: \zeta_0 > 0;$$

(F5) $\kappa := \liminf_{t \to +\infty} \dfrac{tF(t)}{e^{\beta_0 t^2}} > 0.$

Obviously, (F5) is much weaker than (f_8) used in [1, 24]. Our approach is based on delicate estimates for the upper bound for the mountain-pass minimax level c.

Remark 1.1 It is clearly to see that (Q2′) implies (Q2). Since $b < \frac{a(2+\alpha)}{4}$ in (Q1′), it is also easy to verify that (V1) and (Q1′) imply (Q1). However, there are many functions $V(r)$ and $Q(r)$ satisfying (V1), (Q1) and (Q2) but not (Q1′). For example, $V(r) = \log(1+r)$ and $Q(r) = r^\sigma$ satisfy (V1), (Q1) and (Q2) when $\sigma \in \left(-\frac{2+\alpha}{2}, 0\right]$ but not (Q1′) when $\sigma = 0$.

Definition 1.2 We say that u is a least energy solution to (1.1) if $u \in E$ such that $\Phi(u) = m := \inf_{\mathcal{M}} \Phi$, where

$$\mathcal{M} := \{u \in E \setminus \{0\} : \Phi'(u) = 0\}. \tag{1.4}$$

Specifically, we are ready to state the main results in the present paper.

Theorem 1.3 *Assume that V, Q and f satisfy* (V1), (Q1) *and* (F1)-(F3). *Then* (1.1) *has a least energy solution* $u \in E \setminus \{0\}$.

Theorem 1.4 *Assume that V, Q and f satisfy* (V1), (Q1), (Q2), (F1′), (F2), (F3), (F4) *and* (F5). *Then* (1.1) *has a least energy solution* $u \in E \setminus \{0\}$.

Remark 1.5 (F5) is a very mild condition involving the behavior of the nonlinearity f at infinity. Since (F5) is much weaker than (f_8) used in [1, 24], Theorem 1.4 seems to be an innovative result to some extent, which improves and extends the existing results in the direction concerning Choquard equations.

Remark 1.6 We give an explicit examples of nonlinear term satisfying our assumptions (F1), (F2) and (F3) as follows:

$$f_1(t) = \frac{\alpha+2}{2}|t|^{\frac{\alpha-2}{2}}te^{\beta t^2/2} + \beta|t|^{\frac{\alpha+2}{2}}te^{\beta t^2/2} \; (\beta > 0) \text{ and } F_1(t) = \int_0^t f_1(s)\mathrm{d}s = |t|^{\frac{\alpha+2}{2}}e^{\beta t^2/2}.$$

The example of nonlinear term satisfying our assumptions (F1′), (F2), (F3), (F4) and (F5) can be given as follows:

$$f_2(t) = \frac{\alpha+2}{2}|t|^{\frac{\alpha-2}{2}}te^{\beta_0 t^2} + 2\beta_0|t|^{\frac{\alpha+2}{2}}te^{\beta_0 t^2} \; (\beta_0 > 0) \text{ and } F_2(t) = \int_0^t f_2(s)\mathrm{d}s = |t|^{\frac{\alpha+2}{2}}e^{\beta_0 t^2}.$$

When $Q(|x|) = \frac{1}{\sqrt{A_\alpha}|x|^\mu}$ in (1.1), where $0 \le \mu < \frac{\alpha}{2}$, it transforms into the following Schrödinger equation with Stein-Weiss Potential

$$-\Delta u + V(|x|)u = \frac{1}{|x|^\mu}\left(\int_{\mathbb{R}^2} \frac{F(u(y))}{|x-y|^{2-\alpha}|y|^\mu}dy\right)f(u), \text{ in } \mathbb{R}^2. \tag{1.5}$$

There exists $b_0 = -\mu > -\frac{\alpha}{2} > -\frac{2+\alpha}{2}$ such that $(Q1)$ and $(Q2)$ hold.

Corollary 1.7 *Assume that V and f satisfy* (V1) *and* (F1)-(F3). *Then* (1.5) *has a least energy solution* $u \in E \setminus \{0\}$.

Corollary 1.8 *Assume that V and f satisfy* (V1), (F1′), (F2), (F3), (F4) *and* (F5). *Then* (1.5) *has a least energy solution* $u \in E \setminus \{0\}$.

Remark 1.9 When $V(x) \equiv$ constant, there exist $a_0 = 0$ and $a = 0$ such that $(V1)$ holds. Thus Theorem 1.4 and Corollary 1.8 extend and cover the main result in [26].

The paper is organized as follows. In Section 2, we give the variational setting and some preliminary lemmas. Section 3 is devoted to the subcritical exponential growth case where Theorem 1.3 is proved. In Section 4, we consider the critical exponential growth case, and complete the proof of Theorem 1.4.

Throughout the sequel, we denote the usual Lebesgue space with norm $\|u\|_p = \left(\int_{\mathbb{R}^2} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}$ by $L^p(\mathbb{R}^2)$, where $1 \leq p < \infty$, $B_r := \{x \in \mathbb{R}^2 : |x| < r\}$ for all $r > 0$, and C_i denotes different positive constant in different place.

2 Variational framework and preliminaries

Under assumptions $(F1')$ and $(F2)$, fix $\beta > \beta_0$, we know that for any $q \geq 0$, there exist $\theta_1 > 0$, $\theta_{2,q} > 0$, $\theta_{3,q} > 0$ such that

$$|f(t)| \leq \theta_1 |t|^{\frac{\alpha}{2}} + \theta_{2,q}(e^{\beta t^2} - 1)|t|^q, \quad \forall\, t \in \mathbb{R}, \tag{2.1}$$

$$|F(t)| \leq \theta_1 |t|^{\frac{2+\alpha}{2}} + \theta_{2,q}(e^{\beta t^2} - 1)|t|^{q+1}, \quad \forall\, t \in \mathbb{R} \tag{2.2}$$

and

$$|F(t)| \leq \theta_{3,q}(e^{\beta t^2} - 1)|t|^{q+1}, \quad \forall\, |t| \geq 1. \tag{2.3}$$

Similarly, under assumptions $(F1)$ and $(F2)$, we know that (2.1), (2.2) and (2.3) hold for fixed $\beta > 0$.

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality, [27]) *Let $s, r > 1$ and $0 < \alpha < 2$ with $\frac{2-\alpha}{2} + \frac{1}{s} + \frac{1}{r} = 2$, $g \in L^s(\mathbb{R}^2)$, $h \in L^r(\mathbb{R}^2)$. There exists a sharp constant $\mathcal{C}(\alpha, s, r)$, independent of g, h, such that*

$$\int_{\mathbb{R}^2} (I_\alpha * g)h \mathrm{d}x \leq \mathcal{C}(\alpha, s, r)\|g\|_s \|h\|_r. \tag{2.4}$$

In particular,

$$\int_{\mathbb{R}^2} (I_\alpha * g)h \mathrm{d}x \leq \mathcal{C}_0 \|g\|_{4/(2+\alpha)} \|h\|_{4/(2+\alpha)}, \tag{2.5}$$

where $\mathcal{C}_0 := \mathcal{C}(\alpha, 4/(2+\alpha), 4/(2+\alpha))$.

Lemma 2.2 (Cauchy-Schwarz type inequality,[28]) *For $g, h \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, there holds*

$$\int_{\mathbb{R}^2} (I_\alpha * |g|)|h| \mathrm{d}x \leq \left[\int_{\mathbb{R}^2} (I_\alpha * |g|)|g| \mathrm{d}x \int_{\mathbb{R}^2} (I_\alpha * |h|)|h| \mathrm{d}x\right]^{\frac{1}{2}}. \tag{2.6}$$

Lemma 2.3 ([29]) *Assume that $(V1)$ holds. Then for any $r_0 > 0$,*

$$|u(x)| \leq \mathcal{C}_0 \|u\| |x|^{-(a+2)/4}, \quad \forall\, u \in E, \ |x| \geq r_0 > 0. \tag{2.7}$$

In the following lemma, we establish the embeddings $E \hookrightarrow L^s(\mathbb{R}^2, Q)$ for all $s \geq 2$ under conditions $(V1)$ and $(Q1)$ and give a simple proof.

Lemma 2.4 *Assume that* (V1) *and* (Q1) *hold. Then the embeddings* $E \hookrightarrow L^s(\mathbb{R}^2, Q)$ *are continuous and compact for all* $2 \leq s < \infty$. *Therefore there exists* $\gamma_s > 0$ *such that*

$$\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^s \mathrm{d}x \right)^{\frac{1}{s}} \leq \gamma_s \|u\|, \quad \forall\, u \in E, \ 2 \leq s < \infty. \tag{2.8}$$

Proof Fixed $s \in [2, +\infty)$. For any $u \in E$, let $\bar{u}(x) = u(x) - u(x/|x|) := u(x) - u_0$. Then $u_0 \in \mathbb{R}$ and $\bar{u} \in H_0^1(B_1)$. By Poincaré inequality, the Hölder inequality and Sobolev imbedding theorem, one has

$$\int_{B_1} |\bar{u}|^\varrho \mathrm{d}x \leq C(\varrho) \left(\int_{B_1} |\nabla \bar{u}|^2 \mathrm{d}x \right)^{\frac{\varrho}{2}}, \quad \forall\, \varrho \geq 1. \tag{2.9}$$

Since $b_0 > -\frac{2+\alpha}{2}$, we can choose $p > 1$ such that $p \frac{4b_0}{2+\alpha} > -2$. Let $p' := p/(p-1)$. Then it follows from the Hölder inequality and (2.9) that

$$\int_{B_1} |x|^{\frac{4b_0}{2+\alpha}} |\bar{u}|^s \mathrm{d}x \leq \left(\int_{B_1} |x|^{\frac{4pb_0}{2+\alpha}} \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{B_1} |\bar{u}|^{p's} \mathrm{d}x \right)^{\frac{1}{p'}}$$

$$\leq C_1 \left(\frac{2\pi}{2 + \frac{4pb_0}{2+\alpha}} \right)^{\frac{1}{p}} \|\nabla \bar{u}\|_2^s$$

$$= C_1 \left(\frac{2\pi}{2 + \frac{4pb_0}{2+\alpha}} \right)^{\frac{1}{p}} \|\nabla u\|_2^s. \tag{2.10}$$

Hence from (2.10) and Lemma 2.3, we have

$$\int_{B_1} |x|^{\frac{4b_0}{2+\alpha}} |u|^s \mathrm{d}x \leq 2^{s-1} \int_{B_1} |x|^{\frac{4b_0}{2+\alpha}} \left(|\bar{u}|^s + |u_0|^s \right) \mathrm{d}x$$

$$= 2^{s-1} \int_{B_1} |x|^{\frac{4b_0}{2+\alpha}} |\bar{u}|^s \mathrm{d}x + \frac{2^s \pi}{2 + \frac{4pb_0}{2+\alpha}} |u_0|^s \leq C_2 \|u\|^s. \tag{2.11}$$

By using Lemma 2.3, one has

$$\int_{B_1^c} V(|x|) |u|^s \mathrm{d}x \leq C_0^{s-2} \|u\|^{s-2} \int_{B_1^c} V(|x|) u^2 \mathrm{d}x \leq C_3 \|u\|^s. \tag{2.12}$$

Thus, it follows from (V1), (Q1), (2.11) and (2.12) that

$$\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^s \mathrm{d}x = \int_{B_1} Q(|x|)^{\frac{4}{2+\alpha}} |u|^s \mathrm{d}x + \int_{B_1^c} Q(|x|)^{\frac{4}{2+\alpha}} |u|^s \mathrm{d}x$$

$$\leq C_4 \int_{B_1} |x|^{\frac{4b_0}{2+\alpha}} |u|^s \mathrm{d}x + C_5 \int_{B_1^c} V(|x|) |u|^s \mathrm{d}x$$

$$\leq C_6 \|u\|^s, \quad \forall\, u \in E. \tag{2.13}$$

This shows that (2.8) holds, i.e. the embeddings $E \hookrightarrow L^s(\mathbb{R}^2, Q)$ are continuous for all $2 \leq s < \infty$.

Next, we prove that the above embeddings are also compact. Let $\{u_n\} \subset E$ be such that $\|u_n\| \leq C_7$. Without loss of generality, we may assume $u_n \rightharpoonup 0$ in E. We claim $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for all $2 \leq s < \infty$.

For any $\varepsilon > 0$, it follows from (Q1) and Lemma 2.3 that there exists $R_\varepsilon > 1$ such that

$$\int_{B_{R_\varepsilon}^c} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^s \mathrm{d}x \leq \varepsilon \int_{B_{R_\varepsilon}^c} V(|x|)|u_n|^s \mathrm{d}x$$

$$\leq \varepsilon \mathcal{C}_0^{s-2} \|u_n\|^{s-2} \int_{B_{R_\varepsilon}^c} V(|x|)u_n^2 \mathrm{d}x$$

$$\leq \varepsilon \mathcal{C}_0^{s-2} \|u_n\|^s \leq \varepsilon \mathcal{C}_0^{s-2} C_7^s. \tag{2.14}$$

Since $u_n \rightharpoonup 0$, then $u_n \to 0$ in $L_{\mathrm{loc}}^\varrho(\mathbb{R}^2)$ for $\varrho \in [1, +\infty)$. It follows that

$$\int_{B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^s \mathrm{d}x \leq C_8 \int_{B_{R_\varepsilon}} |x|^{\frac{4b_0}{2+\alpha}} |u_n|^s \mathrm{d}x$$

$$\leq C_8 \left(\frac{2\pi R_\varepsilon^{2+\frac{4pb_0}{2+\alpha}}}{2 + \frac{4pb_0}{2+\alpha}} \right)^{\frac{1}{p}} \left(\int_{B_{R_\varepsilon}} |u_n|^{p's} \mathrm{d}x \right)^{\frac{1}{p'}}$$

$$= o(1). \tag{2.15}$$

From (2.14) and (2.15), we can deduce that $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for all $2 \leq s < \infty$ due to the arbitrariness of $\varepsilon > 0$. □

Lemma 2.5 ([25, Theorem 1.2]) *Assume that V and Q satisfy* (V1) *and* (Q1). *Then the following conclusions hold.*

i) *If $\beta > 0$ and $u \in E$, then*

$$\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\beta u^2} - 1 \right) \mathrm{d}x < \infty.$$

ii) *If $0 < \beta < \frac{4\pi(2+\alpha+2b_0)}{2+\alpha}$, then there exists a constant $C > 0$ such that*

$$\sup_{u \in E, \|u\| \leq 1} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\beta u^2} - 1 \right) \mathrm{d}x \leq C.$$

In view of Lemma 2.5, it is easy to see that the energy functional associated to problem (1.1)

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V(|x|)u^2 \right] \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|)F(u)\mathrm{d}x, \ u \in E \tag{2.16}$$

is well defined under (V1), (Q1), (F1) (or (F1$'$)) and (F2), and by using standard arguments, we can show that $\Phi \in C^1(E, \mathbb{R})$ with derivative given by

$$\langle \Phi'(u), \varphi \rangle = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla \varphi + V(|x|)u\varphi] \mathrm{d}x - \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u)\varphi \mathrm{d}x \tag{2.17}$$

for all $\varphi \in E$. Now we say that $u \in E$ is a weak solution to problem (1.1) if, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ it holds that $\langle \Phi'(u), \varphi \rangle = 0$.

Finally, let

$$X := \left\{ u \in L_{\mathrm{loc}}^2(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \ \text{and} \ \int_{\mathbb{R}^2} V(|x|)u^2 \mathrm{d}x < +\infty \right\}.$$

Then X is a Hilbert space when endowed with inner product

$$(u, v)_X = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla v + V(|x|)uv] \mathrm{d}x.$$

Next, inspired by [25], we will show that E is a natural constraint to look for critical points of Φ, namely the critical points of the functional restricted to E are true critical points in X.

Lemma 2.6 *Assume that* (V1), (Q1), (F1) *(or* (F1$'$)*) hold. If $\bar{u}$ is a critical point of Φ restricted to E, then $\bar{u}$ is a critical point of Φ on X.*

Proof We only proof the case where (F1$'$) holds, as the case where (F1) holds is similar. Set

$$T_{\bar{u}}(\varphi) := \int_{\mathbb{R}^2} [\nabla \bar{u} \cdot \nabla \varphi + V(|x|)\bar{u}\varphi] \, \mathrm{d}x - \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi \mathrm{d}x. \quad (2.18)$$

We first prove that $T_{\bar{u}}$ is a bounded linear functional on X. Let $\psi \in \mathcal{C}^\infty([0, +\infty), [0, 1])$ be a cut-off function verifying

$$\psi(r) = 1, \ \forall r \in [0, 1]; \quad \psi(r) = 0, \ \forall r \in [2, +\infty); \quad \text{and} \ |\psi'(r)| \le 2, \ \forall r \in [0, +\infty). \quad (2.19)$$

By (V1), (2.19), Poincaré inequality, the Hölder inequality and Sobolev imbedding theorem, one has

$$\int_{B_1} |\varphi(x)|^s \mathrm{d}x \le \int_{B_2} |\psi(|x|)\varphi(x)|^s \mathrm{d}x$$

$$\le C_{13} \left(\int_{B_2} |\nabla(\psi\varphi)|^2 \mathrm{d}x \right)^{\frac{s}{2}}$$

$$\le C_{14} \left(\int_{B_2} \left(|\varphi \nabla \psi|^2 + |\psi \nabla \varphi|^2 \right) \mathrm{d}x \right)^{\frac{s}{2}}$$

$$\le C_{15} \left(\int_{B_2 \setminus B_1} |\varphi|^2 \mathrm{d}x + \int_{B_2} |\nabla \varphi|^2 \mathrm{d}x \right)^{\frac{s}{2}}$$

$$\le C_{16} \left(\int_{B_2 \setminus B_1} V(|x|)|\varphi|^2 \mathrm{d}x + \int_{B_2} |\nabla \varphi|^2 \mathrm{d}x \right)^{\frac{s}{2}}$$

$$\le C_{16} \|\varphi\|_X^s, \quad \forall s \ge 1, \ \varphi \in X. \quad (2.20)$$

Since $b_0 > -\frac{2+\alpha}{2}$, we can choose $p > 1$ such that $p\frac{4b_0}{2+\alpha} > -2$. Let $p' := p/(p-1)$. Then it follows from (V1), (Q1) and the Hölder inequality that

$$\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |\varphi|^2 \mathrm{d}x = \int_{B_1} Q(|x|)^{\frac{4}{2+\alpha}} |\varphi|^2 \mathrm{d}x + \int_{B_1^c} Q(|x|)^{\frac{4}{2+\alpha}} |\varphi|^2 \mathrm{d}x$$

$$\le C_{17} \int_{B_1} |x|^{\frac{4b_0}{2+\alpha}} |\varphi|^2 \mathrm{d}x + C_{18} \int_{B_1^c} V(|x|)|\varphi|^2 \mathrm{d}x$$

$$\le C_{17} \left(\int_{B_1} |x|^{\frac{4pb_0}{2+\alpha}} \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{B_1} |\varphi|^{2p'} \mathrm{d}x \right)^{\frac{1}{p'}} + C_{18} \int_{B_1^c} V(|x|)|\varphi|^2 \mathrm{d}x$$

$$\le C_{19} \|\varphi\|_X^2, \quad \forall \varphi \in X. \quad (2.21)$$

By (F1$'$), (F2), (2.20), (2.21), Lemmas 2.4 and 2.5, one has

$$\left| \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi \mathrm{d}x \right|$$

$$\leq C_0 \left(\int_{\mathbb{R}^2} |Q(|x|)F(\bar{u})|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}} \left(\int_{\mathbb{R}^2} |Q(|x|)f(\bar{u})\varphi|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{20} \left(\int_{\mathbb{R}^2} \left| Q(|x|)\left[\bar{u}^2 + \left(e^{2\beta_0 \bar{u}^2} - 1\right)|\bar{u}|\right] \right|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}}$$

$$\times \left(\int_{\mathbb{R}^2} \left| Q(|x|)\left[|\bar{u}| + \left(e^{2\beta_0 \bar{u}^2} - 1\right)\right]|\varphi| \right|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{21} \left[\left(\int_{\mathbb{R}^2} \left| Q(|x|)\bar{u}^2 \right|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}} + \left(\int_{\mathbb{R}^2} \left| Q(|x|)\left(e^{2\beta_0 \bar{u}^2} - 1\right)|\bar{u}| \right|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}} \right]$$

$$\times \left[\left(\int_{\mathbb{R}^2} |Q(|x|)\bar{u}\varphi|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}} + \left(\int_{\mathbb{R}^2} \left| Q(|x|)\left(e^{2\beta_0 \bar{u}^2} - 1\right)|\varphi| \right|^{\frac{4}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}} \right]$$

$$\leq C_{22} \left[\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |\bar{u}|^{\frac{8}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{4}} + \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{16\beta_0}{2+\alpha} \bar{u}^2} - 1 \right) \, dx \right)^{\frac{2+\alpha}{8}} \right]$$

$$\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |\bar{u}|^{\frac{8}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{8}} \right] \times \left[\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |\bar{u}|^{\frac{8}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{8}} \right.$$

$$+ \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{16\beta_0}{2+\alpha} \bar{u}^2} - 1 \right) \, dx \right)^{\frac{2+\alpha}{8}} \right] \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |\varphi|^{\frac{8}{2+\alpha}} \, dx \right)^{\frac{2+\alpha}{8}}$$

$$\leq C_{23} \|\varphi\|_X, \quad \forall \, \varphi \in X. \tag{2.22}$$

From (2.18) and (2.22), we obtain

$$|T_{\bar{u}}(\varphi)| \leq |(\bar{u}, \varphi)| + \left| \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi \, dx \right|$$
$$\leq \|\bar{u}\|_X \|\varphi\|_X + C_{23} \|\varphi\|_X \leq C_{24} \|\varphi\|_X, \quad \forall \, \varphi \in X. \tag{2.23}$$

The above shows that $T_{\bar{u}}$ is a bounded linear functional on X. The Riesz Representation Theorem in the Hilbert space X guarantees the existence of a unique $\bar{v} \in X$ such that

$$T_{\bar{u}}(\varphi) = (\varphi, \bar{v})_X, \quad \forall \, \varphi \in X. \tag{2.24}$$

Let $\mathcal{O}(2)$ the group of orthogonal transformations in $\mathbb{R}^2$. Then, by using a change of variables, we get

$$T_{\bar{u}}(g\varphi) = T_{\bar{u}}(\varphi) \text{ and } \|g\varphi\|_X = \|\varphi\|_X, \quad \forall \, \varphi \in X, \, \forall \, g \in \mathcal{O}(2),$$

which, together with (2.24), yields

$$(\varphi, g\bar{v})_X = T_{\bar{u}}(g^{-1}\varphi) = T_{\bar{u}}(\varphi) = (\varphi, \bar{v})_X, \, \forall \, \varphi \in X, \, \forall \, g \in \mathcal{O}(2). \tag{2.25}$$

By uniqueness, one has $g\bar{v} = \bar{v}, \forall \, g \in \mathcal{O}(2)$. This shows that $\bar{v} \in E$. Since $\bar{u}$ is a critical point of Φ restricted to E, it follows that $0 = T_{\bar{u}}(\bar{v}) = \|\bar{v}\|_X^2$. Hence,

$$T_{\bar{u}}(\varphi) = (\varphi, \bar{v})_X = (\varphi, 0)_X = 0, \, \forall \, \varphi \in X,$$

i.e. $\bar{u}$ is a critical point of Φ on X. $\qquad \square$

3 The subcritical case

We establish the same conclusion in Lemma 4.1 below. So, to avoid repetition, we omit the proof of Lemma 3.1 here, which can be deduced obviously from Lemma 4.1.

Lemma 3.1 *Assume that* (V1), (Q1), (F1) *and* (F2) *hold. Then there exists a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{3.1}$$

where c is given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),$$

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1], E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Lemma 3.2 *Assume that* (Q1), (F1) *and* (F2) *hold. Let* $u_n \rightharpoonup \bar{u}$ *in E. Then for every* $v \in E$,

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)v\mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})v\mathrm{d}x. \tag{3.2}$$

Proof Since $u_n \rightharpoonup \bar{u}$ in E, we can choose $C_1 > 0$ such that $\|u_n\| \le C_1$. Set $\beta \in (0, \pi\alpha(2 + \alpha + 2b_0)/(2 + \alpha)C_1^2)$. Then from (2.2), the Hölder inequality and Lemma 2.5-ii), we have

$$\int_{\mathbb{R}^2} |Q(|x|)F(u_n)|^{\frac{4}{2+\alpha}} \mathrm{d}x$$

$$\le \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(C_2|u_n|^{\frac{2+\alpha}{2}} + C_2(e^{\beta u_n^2} - 1)|u_n|\right)^{\frac{4}{2+\alpha}} \mathrm{d}x$$

$$\le 4C_2^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x + 4C_2^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\beta u_n^2} - 1\right)^{\frac{4}{2+\alpha}} |u_n|^{\frac{4}{2+\alpha}} \mathrm{d}x$$

$$\le 4C_2^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x + 4C_2^{\frac{4}{2+\alpha}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1}4\beta u_n^2} - 1\right) \mathrm{d}x\right)^{\frac{\alpha}{2+\alpha}}$$

$$\times \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x\right)^{\frac{2}{2+\alpha}}$$

$$\le 4C_2^{\frac{4}{2+\alpha}} \gamma_2^2 \|u_n\|^2 + 4C_2^{\frac{4}{2+\alpha}} C_3 \gamma_2^2 \|u_n\|^{\frac{4}{2+\alpha}}$$

$$\le 4C_2^{\frac{4}{2+\alpha}} \gamma_2^2 C_1^2 + 4C_2^{\frac{4}{2+\alpha}} C_3 \gamma_2^2 C_1^{\frac{4}{2+\alpha}} =: K_1^{\frac{4}{2+\alpha}}. \tag{3.3}$$

By Lemma 2.4, we know that $v \in L^s(\mathbb{R}^2, Q)$ for $s \ge 2$. And then for any $\varepsilon > 0$, we can choose $R_\varepsilon > 0$ such that

$$\left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} v^2 \mathrm{d}x\right)^{\frac{1}{2}} < \varepsilon. \tag{3.4}$$

From (2.5), (2.1), (3.3), (3.4), the Hölder inequality and Lemma 2.5-ii), one obtains

$$\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)v|\mathrm{d}x$$

$$\le C_0 \left(\int_{\mathbb{R}^2} |Q(|x|)F(u_n)|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}} \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} |Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}}$$

$$\leq C_0 K_1 \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} \left(C_2 |u_n|^{\frac{\alpha}{2}} + C_2 \left(e^{\beta u_n^2} - 1 \right) \right)^{\frac{4}{2+\alpha}} |v|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_4 \left[\left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 dx \right)^{\frac{\alpha}{2+\alpha}} \right.$$

$$\left. + \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1} 4 \beta u_n^2} - 1 \right) dx \right)^{\frac{\alpha}{2+\alpha}} \right]^{\frac{2+\alpha}{4}}$$

$$\times \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2 dx \right)^{\frac{1}{2}}$$

$$\leq C_4 \left(\gamma_2^2 \|u_n\|^{\frac{2\alpha}{2+\alpha}} + C_3 \right)^{\frac{2+\alpha}{4}} \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2 dx \right)^{\frac{1}{2}}$$

$$\leq C_5 \varepsilon. \tag{3.5}$$

Similarly, we can deduce that

$$\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} |[I_\alpha * (Q(|x|) F(\bar{u}))] Q(|x|) f(\bar{u}) v| dx \leq C_6 \varepsilon. \tag{3.6}$$

Since $v \in L^2(\mathbb{R}^2, Q)$, it follows that there exists $\delta_\varepsilon > 0$ such that

$$\left(\int_A Q(|x|)^{\frac{4}{2+\alpha}} v^2 dx \right)^{\frac{1}{2}} < \varepsilon \quad \text{if meas}(A) \leq \delta_\varepsilon \tag{3.7}$$

for all measurable set $A \subset B_{R_\varepsilon}$. Hence it follows from $\|u_n\| \leq C_1$ that there exists $M_\varepsilon > 0$ such that

$$meas(\{x \in B_{R_\varepsilon} : |u_n(x)| \geq M_\varepsilon\}) \leq \delta_\varepsilon, \quad meas(\{x \in B_{R_\varepsilon} : |\bar{u}(x)| \geq M_\varepsilon\}) \leq \delta_\varepsilon. \tag{3.8}$$

Let

$$A_n := \{x \in B_{R_\varepsilon} : |u_n(x)| \geq M_\varepsilon\}, \quad A_0 := \{x \in B_{R_\varepsilon} : |\bar{u}(x)| \geq M_\varepsilon\},$$

$$D_0 := \{x \in B_{R_\varepsilon} : |\bar{u}(x)| = M_\varepsilon\}.$$

Then it follows from (2.5), (2.1), (3.3), (3.7), (3.8), the Hölder inequality and Lemma 2.5-ii) that

$$\int_{A_n \cup D_0} |[I_\alpha * (Q(|x|) F(u_n))] Q(|x|) f(u_n) v| dx$$

$$\leq C_0 \left(\int_{\mathbb{R}^2} |Q(|x|) F(u_n)|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}} \left(\int_{A_n \cup D_0} |Q(|x|) f(u_n) v|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_0 K_1 \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} \left(C_2 |u_n|^{\frac{\alpha}{2}} + C_2 \left(e^{\beta u_n^2} - 1 \right) \right)^{\frac{4}{2+\alpha}} |v|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_7 \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^{\frac{2\alpha}{2+\alpha}} |v|^{\frac{4}{2+\alpha}} dx \right.$$

$$\left. + \int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\beta u_n^2} - 1 \right)^{\frac{4}{2+\alpha}} |v|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_7 \left[\left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x \right)^{\frac{\alpha}{2+\alpha}} \right.$$

$$\left. + \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1}(2+\alpha)\beta u_n^2} - 1 \right) \mathrm{d}x \right)^{\frac{2+\alpha}{4}} \right]^{\frac{2+\alpha}{4}}$$

$$\times \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq C_7 \left(\gamma_2^2 \|u_n\|^{\frac{2\alpha}{2+\alpha}} + C_3 \right)^{\frac{2+\alpha}{4}} \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq C_8 \varepsilon. \tag{3.9}$$

Similarly, we can show that

$$\int_{A_0} |[I_\alpha * (Q(|x|)F(\bar{u}))] Q(|x|) f(\bar{u}) v| \mathrm{d}x \leq C_6 \varepsilon. \tag{3.10}$$

Choose $K_\varepsilon > \max\{1, t_0\}$ such that

$$K_\varepsilon^{-1} \left(\int_{B_{R_\varepsilon}} |Q(|x|) f(u_n) v|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}} < \varepsilon \tag{3.11}$$

and

$$\int_{B_{R_\varepsilon} \setminus A_0} [I_\alpha * (Q(|x|)F(\bar{u})\chi_{|\bar{u}|>K_\varepsilon})] |Q(|x|) f(\bar{u}) v| \mathrm{d}x < \varepsilon. \tag{3.12}$$

Then from (2.5), (2.3) and (3.11) the Hölder inequality and Lemma 2.5-ii), one has

$$\int_{B_{R_\varepsilon} \setminus (A_n \cup D_0)} \left| [I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n| \geq K_\varepsilon})] Q(|x|) f(u_n) v \right| \mathrm{d}x$$

$$\leq C_0 \left(\int_{|u_n| \geq K_\varepsilon} |Q(|x|) F(u_n)|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}} \left(\int_{B_{R_\varepsilon} \setminus (A_n \cup D_0)} |Q(|x|) f(u_n) v|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_9 K_\varepsilon^{-1} \left(\int_{|u_n| \geq K_\varepsilon} \left| Q(|x|) \left(e^{\beta u_n^2} - 1 \right) |u_n|^2 \right|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\left(\int_{B_{R_\varepsilon}} |Q(|x|) f(u_n) v|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{10} K_\varepsilon^{-1} \left(\int_{|u_n| \geq K_\varepsilon} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1}4\beta u_n^2} - 1 \right) \mathrm{d}x \right)^{\frac{\alpha}{4}} \left(\int_{|u_n| \geq K_\varepsilon} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^4 \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\times \left(\int_{B_{R_\varepsilon}} |Q(|x|) f(u_n) v|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{11} K_\varepsilon^{-1} \left(\int_{B_{R_\varepsilon}} |Q(|x|) f(u_n) v|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{11} \varepsilon. \tag{3.13}$$

Let

$$\zeta_n(x) := \frac{1}{A_\alpha}(I_\alpha * (|Q(|x|)F(u_n)|\chi_{|u_n| \le K_\varepsilon}))(x) = \int_{\mathbb{R}^2} \frac{|Q(|y|)F(u_n(y))|\chi_{|u_n(y)| \le K_\varepsilon}}{|x-y|^{2-\alpha}} dy \tag{3.14}$$

and

$$\zeta(x) := \frac{1}{A_\alpha}(I_\alpha * (|Q(|x|)F(\bar{u})|\chi_{|\bar{u}| \le K_\varepsilon}))(x) = \int_{\mathbb{R}^2} \frac{|Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)| \le K_\varepsilon}}{|x-y|^{2-\alpha}} dy. \tag{3.15}$$

Then from (2.2), (3.14) and (3.15), one has

$$|\zeta_n(x) - \zeta(x)|$$

$$\le \int_{\mathbb{R}^2} \frac{\left| |Q(|y|)F(u_n(y))|\chi_{|u_n(y)| \le K_\varepsilon} - |Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)| \le K_\varepsilon} \right|}{|x-y|^{2-\alpha}} dy$$

$$\le \left(\int_{|x-y| \le R} \left| |Q(|y|)F(u_n(y))|\chi_{|u_n(y)| \le K_\varepsilon} - |Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)| \le K_\varepsilon} \right|^{\frac{4-\alpha}{\alpha}} dy \right)^{\frac{\alpha}{4-\alpha}}$$

$$\times \left(\int_{|x-y| \le R} \frac{1}{|x-y|^{(4-\alpha)/2}} dy \right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ \left(\int_{|x-y| > R} \left| |Q(|y|)F(u_n(y))|\chi_{|u_n(y)| \le K_\varepsilon} - |Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)| \le K_\varepsilon} \right|^{\frac{4}{2+\alpha}} dy \right)^{\frac{2+\alpha}{4}}$$

$$\times \left(\int_{|x-y| > R} \frac{1}{|x-y|^4} dy \right)^{\frac{2-\alpha}{4}}$$

$$\le \left(\int_{|x-y| \le R} \left| |Q(|y|)F(u_n(y))|\chi_{|u_n(y)| \le K_\varepsilon} - |Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)| \le K_\varepsilon} \right|^{\frac{4-\alpha}{\alpha}} dy \right)^{\frac{\alpha}{4-\alpha}}$$

$$\left(\frac{4\pi}{\alpha} R^{\frac{\alpha}{2}} \right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ C_\varepsilon \left(\int_{|x-y| > R} Q(|y|)^{\frac{4}{2+\alpha}} |u_n|^2 dy + \int_{|x-y| > R} Q(|y|)^{\frac{4}{2+\alpha}} |\bar{u}|^2 dy \right)^{\frac{2+\alpha}{4}} \left(\frac{\pi}{R^2} \right)^{\frac{2-\alpha}{4}}$$

$$\le \left(\frac{4\pi}{\alpha} R^{\frac{\alpha}{2}} \right)^{\frac{4-2\alpha}{4-\alpha}} o(1) + C_{12} C_\varepsilon \left(\frac{\pi}{R^2} \right)^{\frac{2-\alpha}{4}}, \ \forall x \in \mathbb{R}^2, \tag{3.16}$$

which implies that

$$\zeta_n(x) \to \zeta(x), \ \forall x \in \mathbb{R}^2. \tag{3.17}$$

From (2.2) and (3.14), we have

$$|\zeta_n(x)| \le \int_{\mathbb{R}^2} \frac{\left| |Q(|y|)F(u_n(y))|\chi_{|u_n(y)| \le K_\varepsilon} \right|}{|x-y|^{2-\alpha}} dy$$

$$\le \left(\int_{|x-y| \le 1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)| \le K_\varepsilon}|^{\frac{4-\alpha}{\alpha}} dy \right)^{\frac{\alpha}{4-\alpha}}$$

$$\left(\int_{|x-y| \le 1} \frac{1}{|x-y|^{(4-\alpha)/2}} dy \right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ \left(\int_{|x-y|>1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4}{2+\alpha}} dy \right)^{\frac{2+\alpha}{4}} \left(\int_{|x-y|>1} \frac{1}{|x-y|^4} dy \right)^{\frac{2-\alpha}{4}}$$

$$= \left(\int_{|x-y|\leq 1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4-\alpha}{\alpha}} dy \right)^{\frac{\alpha}{4-\alpha}} \left(\frac{4\pi}{\alpha} \right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ \left(\int_{|x-y|>1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4}{2+\alpha}} dy \right)^{\frac{2+\alpha}{4}} \pi^{\frac{2-\alpha}{4}}$$

$$\leq \max_{B(x,1)} |Q(|y|)| \max_{|t|\leq K_\varepsilon} |F(t)| \pi^{\frac{\alpha}{4-\alpha}} \left(\frac{4\pi}{\alpha} \right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ C_\varepsilon \left(\int_{|x-y|>1} Q(|y|)^{\frac{4}{2+\alpha}} |u_n|^2 \right)^{\frac{2+\alpha}{4}} \pi^{\frac{2-\alpha}{4}}$$

$$\leq C_{13} \max_{B(x,1)} |Q(|y|)| + C_{14}C_\varepsilon, \ \forall \, x \in \mathbb{R}^2. \tag{3.18}$$

It follows that

$$\left| \zeta_n(x)Q(|x|)f(u_n(x))\chi_{|u_n(x)|\leq M_\varepsilon} v(x) \right|$$
$$\leq \left[\left(C_{13} \max_{y \in B_{R_\varepsilon+1}} |Q(|y|)| + C_{14} \right) \max_{x \in B_{R_\varepsilon}} |Q(|x|)| \max_{|t|\leq M_\varepsilon} |f(t)| \right] |v(x)| \in L^1(B_{R_\varepsilon}),$$
$$\forall \, x \in B_{R_\varepsilon}. \tag{3.19}$$

Since $u_n \rightharpoonup \bar{u}$ in E, we can deduce that $u_n \to \bar{u}$ a.e. $x \in \mathbb{R}^2$. By (3.17), we can deduce that

$$\zeta_n(x)Q(|x|)f(u_n(x))\chi_{|u_n(x)|\leq M_\varepsilon} v(x) \to \zeta(x)Q(|x|)f(\bar{u}(x))\chi_{|\bar{u}(x)|\leq M_\varepsilon} v(x) \ a.e. \ x \in B_{R_\varepsilon}\backslash D_0.$$

Therefore, (3.17), together with (3.19) and Lebesgue dominated convergence theorem lead to

$$\lim_{n\to\infty} \int_{B_{R_\varepsilon}\backslash(A_n\cup D_0)} [I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n|\leq K_\varepsilon})]Q(|x|)f(u_n)v\,dx$$
$$= \int_{B_{R_\varepsilon}\backslash A_0} [I_\alpha * (Q(|x|)F(\bar{u})\chi_{|\bar{u}|\leq K_\varepsilon})]Q(|x|)f(\bar{u})v\,dx. \tag{3.20}$$

It follows from (3.5), (3.6), (3.9), (3.10), (3.12), (3.13) and (3.20) that (3.2) holds due to the arbitrariness of $\varepsilon > 0$. $\qquad\square$

By a standard argument, we can get the following lemma.

Lemma 3.3 *Assume that* (V1), (Q1) *and* (F1)-(F3) *hold. Then any sequence* $\{u_n\}$ *satisfying* (3.1) *is bounded in* E.

Lemma 3.4 *Assume that* (V1), (Q1) *and* (F1)-(F3) *hold. Then* Φ *satisfies* $(C)_c$ *condition.*

Proof Applying Lemmas 3.1 and 3.3, we deduce that there exists a sequence $\{u_n\} \subset E$ satisfying (3.1) and $\|u_n\| \leq C_1$ for some constant $C_1 > 0$. Since $\|u_n\| \leq C_1$, by Lemma 2.4, we may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2, Q)$ for $s \in [2, \infty)$, and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. By (2.16) and (3.1), we can deduce that

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n) \leq C_{15}. \tag{3.21}$$

Hence it follows from (3.21), (2.3), Lemma 2.5-ii), $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and Hölder inequality that

$$\int_{\mathbb{R}^2} \left| [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u}) \right| dx$$

$$\leq \left(\int_{\mathbb{R}^2} \left| [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n) \right| dx \right)^{\frac{1}{2}}$$

$$\times \left(\int_{\mathbb{R}^2} \left| [I_\alpha * (Q(|x|)f(u_n)(u_n - \bar{u}))]Q(|x|)f(u_n)(u_n - \bar{u}) \right| dx \right)^{\frac{1}{2}}$$

$$\leq \sqrt{C_{15}} \left(\int_{\mathbb{R}^2} \left| Q(|x|)f(u_n)(u_n - \bar{u}) \right|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{16} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 dx \right)^{\frac{4\alpha}{(2+\alpha)^2}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^2 dx \right)^{\frac{8}{(2+\alpha)^2}}$$

$$+ C_{16} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{8\beta}{2+\alpha}u_n^2} - 1 \right) dx \right)^{\frac{2+\alpha}{8}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^{\frac{8}{2+\alpha}} dx \right)^{\frac{2+\alpha}{8}}$$

$$\leq C_{17} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^2 dx \right)^{\frac{8}{(2+\alpha)^2}} + C_{17} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^{\frac{8}{2+\alpha}} dx \right)^{\frac{2+\alpha}{8}}$$

$$= o(1). \tag{3.22}$$

Which yields that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u}) dx = 0. \tag{3.23}$$

Similarly, we can deduce that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})(u_n - \bar{u}) dx = 0. \tag{3.24}$$

By (3.1), it is obvious that

$$\langle \Phi'(u_n) - \Phi'(\bar{u}), u_n - \bar{u} \rangle = o(1). \tag{3.25}$$

From (3.23), (3.24) and (3.25), we can get

$$\|u_n - \bar{u}\|^2 = \langle \Phi'(u_n) - \Phi'(\bar{u}), u_n - \bar{u} \rangle - \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u}) dx$$

$$+ \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})(u_n - \bar{u}) dx$$

$$= o(1).$$

The following lemma allows us to find a least energy solution for (1.1) with positive level.

Lemma 3.5 *Assume that* (V1), (Q1), (F1), (F2) *and* (F3) *hold. Then*

i) $\sigma_0 := \inf\{\|u\| : u \in \mathcal{M}\} > 0$;
ii) $m := \inf\{\Phi(u) : u \in \mathcal{M}\} > 0$.

Proof (i) Choose $\beta \in (0, \pi(2 + \alpha + b_0)/2)$. For any $u \in \mathcal{M}$, if $\|u\| < 1$, by (2.1) and (2.2), we have

$$\|u\|^2 = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u)u\,dx$$

$$\leq \left(\int_{\mathbb{R}^2} |Q(|x|)F(u)|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}} \left(\int_{\mathbb{R}^2} |Q(|x|)f(u)u|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{18} \left[\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(|u|^2 + \left(e^{\beta u^2} - 1 \right)^{\frac{4}{2+\alpha}} |u|^2 \right) \mathrm{d}x \right]^{\frac{2+\alpha}{2}}$$

$$\leq C_{19} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^2 \mathrm{d}x \right)^{\frac{2+\alpha}{2}}$$

$$+ C_{19} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{(2+\alpha)^{-1} 8\beta u^2} - 1 \right) \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\times \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^4 \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{20} \|u\|^{2+\alpha} + C_{21} \|u\|^{2+\alpha} = C_{22} \|u\|^{2+\alpha}. \tag{3.26}$$

Thus there exists $C_{23} > 0$ such that $\|u\| \geq C_{23}$, $\forall u \in \mathcal{M}$, $\|u\| < 1$. Taking $\sigma_0 = \min\{C_{23}, 1\}$, then $\|u\| \geq \sigma_0$, $\forall u \in \mathcal{M}$.

(ii) For any $u \in \mathcal{M}$, by (F3) and item i), it is easy to see that

$$\Phi(u) = \Phi(u) - \frac{1}{2\mu} \langle \Phi'(u), u \rangle \geq \frac{\mu - 1}{2\mu} \|u\|^2 \geq \frac{\mu - 1}{2\mu} \sigma_0^2.$$

Thus we can deduce that item ii) holds.

Proof of Theorem 1.3 First, we prove that $\mathcal{M} \neq \emptyset$. Applying Lemmas 3.1 and 3.2, we deduce that there exists a sequence $\{v_n\} \subset E$ satisfying (3.1) and $\|v_n\| \leq C_{24}$ for some constant $C_{24} > 0$. Hence, there exists $v \in E$ such that, up to a subsequence, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s(\mathbb{R}^2)$ for $s \geq 2$ and $v_n \to v$ a.e. in $\mathbb{R}^2$. By Lemma 3.4, we know that $v_n \to v$ in E. Thus

$$\Phi(v) = \lim_{n \to \infty} \Phi(v_n) = c > 0, \quad \Phi'(v) = \lim_{n \to \infty} \Phi'(v_n) = 0.$$

Which imply that $v \neq 0$ and $v \in \mathcal{M}$.

Next, we prove that $m := \inf_{\mathcal{M}} \Phi(u)$ is achieved. By Lemma 3.5, we have $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. It is easy to check that

$$m + o(1) = \Phi(u_n) = \Phi(u_n) - \frac{1}{2\mu} \langle \Phi'(u_n), u_n \rangle$$

$$= \frac{\mu - 1}{2\mu} \|u_n\|^2 + \frac{1}{2\mu} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))] Q(|x|) [f(u_n)u_n - \mu F(u_n)] \, \mathrm{d}x$$

$$\geq \frac{\mu - 1}{2\mu} \|u_n\|^2.$$

Which imply that $\{u_n\}$ is bounded in E. Hence, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and $u_n \to u$ a.e. in $\mathbb{R}^2$. As in the proof of Lemma 3.4, we can deduce that $u_n \to u$ in E. Thus

$$\Phi(u) = \lim_{n \to \infty} \Phi(u_n) = m > 0, \quad \Phi'(u) = \lim_{n \to \infty} \Phi'(u_n) = 0.$$

$\square$

4 The critical case

Lemma 4.1 *Assume that* (V1), (Q1), (F1$'$) *and* (F2) *hold. Then there exists a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{4.1}$$

where c *is given by*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),$$

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Proof By (F1$'$) and (F2), there exists $C_1 > 0$, $C_2 > 0$ such that

$$|F(t)|^{\frac{4}{2+\alpha}} \le C_1 t^2 + C_2 |t|^2 \left(e^{\frac{8\beta_0}{2+\alpha}t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{4.2}$$

In view of Lemma 2.5-ii), we have

$$\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{16\beta_0}{2+\alpha}u^2} - 1\right) dx = \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{16\beta_0}{2+\alpha}\|u\|^2 (u/\|u\|)^2} - 1\right) dx$$

$$\le C_3, \quad \forall\, \|u\| \le \sqrt{\pi(2 + \alpha + 2b_0)/5\beta_0}. \tag{4.3}$$

From (2.8), (4.2) and (4.3), we obtain

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|)F(u)dx$$

$$\le \left(\int_{\mathbb{R}^2} |Q(|x|)F(u)|^{\frac{4}{2+\alpha}} dx\right)^{\frac{2+\alpha}{2}}$$

$$\le \left[C_1 \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} u^2 dx + C_2 \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^2 \left(e^{\frac{8\beta_0}{2+\alpha}u^2} - 1\right) dx\right]^{\frac{2+\alpha}{2}}$$

$$\le C_4 \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} u^2 dx\right)^{\frac{2+\alpha}{2}}$$

$$+ C_5 \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^4 dx\right)^{\frac{2+\alpha}{4}} \left[\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{16\beta_0}{2+\alpha}u^2} - 1\right) dx\right]^{\frac{2+\alpha}{4}}$$

$$\le C_6 \|u\|^{2+\alpha}, \quad \forall\, \|u\| \le \sqrt{\pi(2 + \alpha + 2b_0)/5\beta_0}. \tag{4.4}$$

Hence, it follows from (2.16) and (4.4) that

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|)F(u)dx$$

$$\ge \frac{1}{2}\|u\|^2 - C_6\|u\|^{2+\alpha}, \quad \forall\, \|u\| \le \sqrt{\pi(2 + \alpha + 2b_0)/5\beta_0}. \tag{4.5}$$

Therefore, there exist $\kappa_0 > 0$ and $0 < \rho_0 < \sqrt{\pi(2 + \alpha + 2b_0)/5\beta_0}$ such that

$$\Phi(u) \ge \kappa_0, \quad \forall\, u \in S := \{u \in E : \|u\| = \rho_0\}. \tag{4.6}$$

Now we choose $w_0 \in E \setminus \{0\}$, it is easy to show that $\lim_{t \to \infty} \Phi(tw_0) = -\infty$ due to (F3). Hence, we can choose $T > 0$ such that $e := Tw_0 \in \{u \in E : \|u\| > \rho_0\}$ and $\Phi(e) < 0$, then in view of the mountain-pass lemma, we deduce that there exists a sequence $\{u_n\} \subset E$ satisfying (4.1). $\qquad\square$

Lemma 4.2 *Assume that* (V1), (Q1), (F1′), (F2) *and* (F3) *hold. Then any sequence* $\{u_n\}$ *satisfying* (4.1) *is bounded in* E.

Lemma 4.3 *Assume that* (Q1), (F1′), (F2), (F3), (F4) *hold. Let* $u_n \rightharpoonup \bar{u}$ *in* E *and*

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n \mathrm{d}x \leq K_0 \tag{4.7}$$

for some constant $K_0 > 0$. *Then there hold:*

i) *for every* $\varphi \in E \cap \mathcal{C}_0^\infty(\mathbb{R}^2)$

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi \mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi \mathrm{d}x; \tag{4.8}$$

ii)

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha*(Q(|x|)F(u_n))]Q(|x|)F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)F(\bar{u})\mathrm{d}x. \tag{4.9}$$

Proof The proof of item ii) is similar to [1, Claim 6.3], so we omit the details here. Next, inspired by [23, Lemma 4.8], we prove i). By the Fatou's lemma and (4.7), we have

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\bar{u}\mathrm{d}x \leq K_0. \tag{4.10}$$

Let $\Omega = \mathrm{supp}\varphi$. For any given $\varepsilon > 0$, let $M_\varepsilon := K_0\|\varphi\|_\infty \varepsilon^{-1}$. Then it follows from (F4), (4.7) and (4.10) that for n large,

$$\int_{|u_n|\geq M_\varepsilon} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi|\mathrm{d}x$$
$$\leq \frac{\|\varphi\|_\infty}{M_\varepsilon} \int_{|u_n|\geq M_\varepsilon} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n|\mathrm{d}x$$
$$\leq \frac{\|\varphi\|_\infty}{M_\varepsilon} K_0 = \varepsilon \tag{4.11}$$

and

$$\int_{|\bar{u}|\geq M_\varepsilon} |[I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi|\mathrm{d}x$$
$$\leq \frac{\|\varphi\|_\infty}{M_\varepsilon} \int_{|\bar{u}|\geq M_\varepsilon} |[I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\bar{u}|\mathrm{d}x$$
$$\leq \frac{\|\varphi\|_\infty}{M_\varepsilon} K_0 = \varepsilon. \tag{4.12}$$

Let $D_\varepsilon = \{x \in \Omega : |\bar{u}(x)| = M_\varepsilon\}$. Since $|Q(|x|)f(u_n)\chi_{|u_n|\leq M_\varepsilon}| \to |Q(|x|)f(\bar{u})\chi_{|\bar{u}|\leq M_\varepsilon}|$ a.e. in $\Omega \setminus D_\varepsilon$ and

$$|Q(|x|)f(u_n)\chi_{|u_n|\leq M_\varepsilon}| \leq \max_{x\in\Omega} Q(|x|) \max_{|t|\leq M_\varepsilon} |f(t)| < \infty, \quad \forall\, x \in \Omega,$$

the Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty} \int_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}} |Q(|x|)f(u_n)|^{\frac{4}{2+\alpha}}\mathrm{d}x = \int_{(\Omega\setminus D_\varepsilon)\cap\{|\bar{u}|\leq M_\varepsilon\}} |Q(|x|)f(\bar{u})|^{\frac{4}{2+\alpha}}\mathrm{d}x.$$

$$(4.13)$$

Choose $K_\varepsilon > t_0$ such that

$$\|\varphi\|_\infty \mathcal{C}_0 \left(\frac{M_0 K_0}{K_\varepsilon} \right)^{\frac{1}{2}} \left(\int_{(\Omega \setminus D_\varepsilon) \cap \{|\bar{u}| \leq M_\varepsilon\}} |Q(|x|) f(\bar{u})|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}} < \varepsilon \qquad (4.14)$$

and

$$\int_{|\bar{u}| < M_\varepsilon} [I_\alpha * (Q(|x|) F(\bar{u}) \chi_{|\bar{u}| > K_\varepsilon})] |Q(|x|) f(\bar{u}) \varphi| \mathrm{d}x < \varepsilon. \qquad (4.15)$$

Then from (F4), (4.13) and (4.14), one has

$$\int_{(|u_n| \leq M_\varepsilon) \cap (|\bar{u}| \neq M_\varepsilon)} \left| [I_\alpha * (Q(|x|) F(u_n) \chi_{|u_n| \geq K_\varepsilon})] Q(|x|) f(u_n) \varphi \right| \mathrm{d}x$$

$$\leq \|\varphi\|_\infty \int_{\Omega \setminus D_\varepsilon} \left| [I_\alpha * (Q(|x|) F(u_n) \chi_{|u_n| \geq K_\varepsilon})] Q(|x|) f(u_n) \chi_{|u_n| \leq M_\varepsilon} \right| \mathrm{d}x$$

$$\leq \|\varphi\|_\infty \left(\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|) F(u_n) \chi_{|u_n| \geq K_\varepsilon})] Q(|x|) F(u_n) \chi_{|u_n| \geq K_\varepsilon} \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\times \left(\int_{\mathbb{R}^2} [I_\alpha * (|Q(|x|) f(u_n)| \chi_{(\Omega \setminus D_\varepsilon) \cap \{|u_n| \leq M_\varepsilon\}})] |Q(|x|) f(u_n)| \chi_{(\Omega \setminus D_\varepsilon) \cap \{|u_n| \leq M_\varepsilon\}} \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq \|\varphi\|_\infty \mathcal{C}_0 \left(\int_{|u_n| \geq K_\varepsilon} [I_\alpha * (Q(|x|) F(u_n))] Q(|x|) F(u_n) \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\times \left(\int_{\mathbb{R}^2} \left| Q(|x|) f(u_n) \chi_{(\Omega \setminus D_\varepsilon) \cap \{|u_n| \leq M_\varepsilon\}} \right|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq \|\varphi\|_\infty \mathcal{C}_0 \left(\frac{M_0}{K_\varepsilon} \int_{|u_n| \geq K_\varepsilon} [I_\alpha * (Q(|x|) F(u_n))] Q(|x|) f(u_n) u_n \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\times \left(\int_{(\Omega \setminus D_\varepsilon) \cap \{|\bar{u}| \leq M_\varepsilon\}} |Q(|x|) f(u_n)|^{\frac{4}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{4}}$$

$$\leq \|\varphi\|_\infty \mathcal{C}_0 \left(\frac{M_0 K_0}{K_\varepsilon} \right)^{\frac{1}{2}} \left(\int_{(\Omega \setminus D_\varepsilon) \cap \{|\bar{u}| \leq M_\varepsilon\}} |Q(|x|) f(\bar{u})|^{\frac{4}{2+\alpha}} \mathrm{d}x + o(1) \right)^{\frac{2+\alpha}{4}}$$

$$< \varepsilon + o(1). \qquad (4.16)$$

For any $x \in \mathbb{R}^2$, we define $\zeta_n(x)$ and $\zeta(x)$ as in (3.14) and (3.15), then from (F1$'$), (F2), (3.16) and (3.18), we also have

$$\zeta_n(x) \to \zeta(x), \quad \forall x \in \mathbb{R}^2 \qquad (4.17)$$

and

$$|\zeta_n(x)| \leq C_7 \max_{B(x,1)} |Q(|y|)| + C_8, \quad \forall x \in \mathbb{R}^2. \qquad (4.18)$$

It follows that

$$\left| \zeta_n(x) Q(|x|) f(u_n(x)) \chi_{|u_n| \leq M_\varepsilon}(x) \varphi(x) \right|$$

$$\leq \left(C_7 \max_{B(x,1)} |Q(|y|)| + C_8 \right) \|\varphi\|_\infty \max_{x \in \Omega} |Q(|x|)| \max_{|t| \leq M_\varepsilon} |f(t)| < \infty, \quad \forall x \in \Omega. \quad (4.19)$$

Since $u_n \rightharpoonup \bar{u}$ in E, we can deduce that $u_n \to \bar{u}$ a.e. $x \in \mathbb{R}^2$. By (4.17), we can deduce that

$$\zeta_n(x)Q(|x|)f(u_n(x))\chi_{|u_n|\leq M_\varepsilon}(x)\varphi(x) \to \zeta(x)Q(|x|)f(\bar{u}(x))\chi_{|\bar{u}|\leq M_\varepsilon}(x)\varphi(x) \quad a.e. x \in \Omega \backslash D_\varepsilon.$$

Therefore, (4.19) and Lebesgue dominated convergence theorem lead to

$$\lim_{n\to\infty} \int_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|\neq M_\varepsilon)} [I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n|\leq K_\varepsilon})]Q(|x|)f(u_n)\varphi \mathrm{d}x$$
$$= \int_{|\bar{u}|<M_\varepsilon} [I_\alpha * (Q(|x|)F(\bar{u})\chi_{|\bar{u}|\leq K_\varepsilon})]Q(|x|)f(\bar{u})\varphi \mathrm{d}x. \tag{4.20}$$

Since $u_n \rightharpoonup \bar{u}$ in E, we can deduce that $u_n \to \bar{u}$ a.e. $x \in \mathbb{R}^2$ and then, by $E_{\Gamma\text{opo}B}$ theorem, we have $u_n \rightrightarrows \bar{u}$, $x \in \Omega \backslash A$, where $m(A) < \varepsilon$. For large n, one has

$$\{x \in \Omega \backslash A : |\bar{u}(x)| = M_\varepsilon\} \subset \left\{x \in \Omega \backslash A : |u_n(x)| > \frac{M_\varepsilon}{2}\right\}.$$

And then from Lemmas 2.1 and 2.2, we obtain

$$\int_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon)} \left|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi\right| \mathrm{d}x$$

$$= \int_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon))\cap(\Omega\backslash A)} \left|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi\right| \mathrm{d}x$$

$$+ \int_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon)\cap A} \left|[I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n|\leq K_\varepsilon})]Q(|x|)f(u_n)\varphi\right| \mathrm{d}x$$

$$+ \int_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon)\cap A} \left|[I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n|>K_\varepsilon})]Q(|x|)f(u_n)\varphi\right| \mathrm{d}x$$

$$\leq \|\varphi\|_\infty \int_{(|u_n|>M_\varepsilon/2)\cap(\Omega\backslash A)} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)| \mathrm{d}x$$

$$+ \|\varphi\|_\infty C_0 \left(\int_{\mathbb{R}^2} \left|Q(|x|)F(u_n)\chi_{|u_n|\leq K_\varepsilon}\right|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}}$$

$$\left(\int_{\mathbb{R}^2} \left|Q(|x|)f(u_n)\chi_{(|u_n|\leq M_\varepsilon)\cap A}\right|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}}$$

$$+ \|\varphi\|_\infty \left(\int_{\mathbb{R}^2} \left|[I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n|>K_\varepsilon})]Q(|x|)F(u_n)\chi_{|u_n|>K_\varepsilon}\right| \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\times \left(\int_{\mathbb{R}^2} \left|[I_\alpha * (Q(|x|)f(u_n)\chi_{(|u_n|\leq M_\varepsilon)\cap A})]Q(|x|)f(u_n)\chi_{(|u_n|\leq M_\varepsilon)\cap A}\right| \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\leq \frac{2\|\varphi\|_\infty}{M_\varepsilon} \int_{|u_n|>M_\varepsilon/2} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n| \mathrm{d}x$$

$$+ C_\varepsilon \|\varphi\|_\infty C_0 \left(\int_{|u_n|\leq K_\varepsilon} Q(|x|)^{\frac{4}{2+\alpha}} u_n^2 \mathrm{d}x\right)^{\frac{2+\alpha}{4}} \left(\int_{(|u_n|\leq M_\varepsilon)\cap A} \left|Q(|x|)f(u_n)\right|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}}$$

$$+ C_\varepsilon \sqrt{\frac{M_0}{K_\varepsilon}} \sqrt{C_0} \left(\int_{|u_n|>K_\varepsilon} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n| \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\times \left(\int_{(|u_n|\leq M_\varepsilon)\cap A} \left|Q(|x|)f(u_n)\right|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}}$$

$$\leq C_9 \varepsilon. \tag{4.21}$$

It follows from (4.11), (4.12), (4.15), (4.16), (4.20) and (4.21) that (4.8) holds due to the arbitrariness of $\varepsilon > 0$. □

As in [30], we define Moser type functions $w_n(x)$ supported in $B_d := B_d(0)$ as follows:

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq d/n; \\ \frac{\log(d/|x|)}{\sqrt{\log n}}, & d/n \leq |x| \leq d; \\ 0, & |x| \geq d. \end{cases} \tag{4.22}$$

By an elemental computation, we have

$$\|\nabla w_n\|^2 = \int_{B_d \setminus B_{d/n}} |\nabla w_n|^2 dx = 1 \tag{4.23}$$

and

$$\int_{\mathbb{R}^2} |x|^{a_0} w_n^2 dx = \frac{2d^{2+a_0}}{(2+a_0)^2} \delta_n, \tag{4.24}$$

where

$$\delta_n := \frac{1}{(2+a_0)\log n} - \frac{1}{(2+a_0)n^{2+a_0}\log n} - \frac{1}{n^{2+a_0}} > 0. \tag{4.25}$$

Lemma 4.4 *Assume that* (V1), (Q1), (Q2) (F1′) *and* (F2)-(F5) *hold. Then there exists $\bar{n} \in \mathbb{N}$ such that*

$$c \leq \max_{t \geq 0} \Phi(t w_{\bar{n}}) < \frac{(2+\alpha+2b_0)\pi}{2\beta_0}. \tag{4.26}$$

Proof By (V1), we can deduce that there exists $d > 0$ such that

$$V(r) \leq (\tau+1)r^{a_0}, \quad \forall \, 0 < r \leq d \tag{4.27}$$

and

$$\frac{(2+\alpha+2b_0)(\tau+1)d^{2+a_0}}{(2+a_0)^3} < \frac{1}{3}. \tag{4.28}$$

By (F5), we know that there exists $t_\kappa > 0$ such that

$$tF(t) \geq \frac{\kappa}{2}e^{\beta_0 t^2}, \quad \forall \, t \geq t_\kappa. \tag{4.29}$$

From (V1), (F5), (2.16), (4.23), (4.24) and (4.27), we have

$$\begin{aligned} \Phi(t w_n) &= \frac{t^2}{2}\left(\|\nabla w_n\|_2^2 + \int_{\mathbb{R}^2} V(|x|)w_n^2 dx\right) \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(tw_n))]Q(|x|)F(tw_n)dx \\ &\leq \frac{t^2}{2}\left(\|\nabla w_n\|_2^2 + \int_{B_d} (\tau+1)|x|^{a_0} w_n^2 dx\right) \\ &\quad - \frac{1}{2}\int_{B_{d/n}} [I_\alpha * (Q(|x|)F(tw_n))]Q(|x|)F(tw_n)dx \\ &\leq \frac{t^2}{2}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right] \end{aligned}$$

$$-\frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2. \tag{4.30}$$

To show (4.26) we have three cases to distinguish. From now on, in the sequel, all inequalities hold for large $n \in \mathbb{N}$.

Case i). $t \in \left[0, \sqrt{\frac{(2+\alpha+2b_0)\pi}{2\beta_0}}\right]$. Then it follows from (F5), (Q2) and (4.30) that

$$\Phi(t w_n) \le \frac{t^2}{2}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right] \le \frac{(2+\alpha+2b_0)\pi}{3\beta_0}. \tag{4.31}$$

Clearly, there exists $\bar{n} \in \mathbb{N}$ such that (4.26) holds.

Case ii). $t \in \left[\sqrt{\frac{(2+\alpha+2b_0)\pi}{2\beta_0}}, \sqrt{\frac{(2+\alpha+2b_0)(1+\kappa)\pi}{\beta_0}}\right]$. Then $t\sqrt{\log n}/\sqrt{2\pi} \ge t_\kappa$ for large $n \in \mathbb{N}$, it follows from (Q2) and (4.29) that

$$\frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2$$

$$\ge \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\frac{2\pi}{t^2\log n}\frac{\kappa^2}{4}e^{\pi^{-1}\beta_0 t^2 \log n}\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2$$

$$\ge \frac{\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\frac{\kappa^2\beta_0}{(2+\alpha+2b_0)(1+\kappa)\log n}e^{\pi^{-1}\beta_0 t^2 \log n}\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2$$

$$\ge \frac{\pi^2\kappa^2\beta_0\zeta_0^2 d^{2+\alpha+2b_0}}{2^{4-\alpha}(2+\alpha+2b_0)(1+\kappa)n^{2+\alpha+2b_0}\log n}e^{\pi^{-1}\beta_0 t^2 \log n}. \tag{4.32}$$

It follows from (4.25), (4.30) and (4.32) that

$$\Phi(t w_n) \le \frac{t^2}{2}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right] - \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2$$

$$\le \frac{t^2}{2}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right]$$

$$\qquad - \frac{\pi^2\kappa^2\beta_0\zeta_0^2 d^{2+\alpha+2b_0}}{2^{4-\alpha}(2+\alpha+2b_0)(1+\kappa)n^{2+\alpha+2b_0}\log n}e^{\pi^{-1}\beta_0 t^2 \log n}$$

$$=: \psi_n(t). \tag{4.33}$$

Let $t_n > 0$ such that $\psi_n'(t_n) = 0$. Then

$$1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n} = \frac{\pi\kappa^2\beta_0^2\zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)n^{2+\alpha+2b_0}}e^{\pi^{-1}\beta_0 t^2 \log n}. \tag{4.34}$$

It follows that

$$t_n^2 = \frac{(2+\alpha+2b_0)\pi}{\beta_0} - \frac{\pi}{\beta_0\log n}\log\frac{\pi\kappa^2\beta_0^2\zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)} + O\left(\frac{1}{\log^2 n}\right) \tag{4.35}$$

It follows from (4.28), (4.33) and (4.35), we have

$$\psi_n(t) \le \psi_n(t_n) = \frac{t_n^2}{2}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right] - \frac{\pi}{2\beta_0\log n}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right]$$

$$= \frac{1}{2}\left[\frac{(2+\alpha+2b_0)\pi}{\beta_0} - \frac{\pi}{\beta_0\log n}\log\frac{\pi\kappa^2\beta_0^2\zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)}\right]$$

$$\times \left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3 \log n} \right] - \frac{\pi}{2\beta_0 \log n} + O\left(\frac{1}{\log^2 n} \right)$$

$$= \frac{(2+\alpha+2b_0)\pi}{2\beta_0} + \frac{\pi}{\beta_0 \log n} \left[\frac{(2+\alpha+2b_0)(\tau+1)d^{2+a_0}}{(2+a_0)^3} - \frac{1}{2} \right]$$

$$- \frac{\pi}{2\beta_0 \log n} \log \frac{\pi \kappa^2 \beta_0^2 \zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)} + O\left(\frac{1}{\log^2 n} \right)$$

$$< \frac{(2+\alpha+2b_0)\pi}{2\beta_0} - \frac{\pi}{2\beta_0 \log n} \log \frac{\pi \kappa^2 \beta_0^2 \zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)} + O\left(\frac{1}{\log^2 n} \right). \tag{4.36}$$

Hence, combining (4.33) with (4.36), one has

$$\Phi(t w_n) \le \psi_n(t) \le \psi_n(t_n) < \frac{(2+\alpha+2b_0)\pi}{2\beta_0}. \tag{4.37}$$

Case iii). $t \in \left(\sqrt{\frac{(2+\alpha+2b_0)\pi}{\beta_0}(1+\kappa)}, +\infty \right)$. Then $t\sqrt{\log n}/\sqrt{2\pi} \ge t_\kappa$ for large $n \in \mathbb{N}$, it follows from (4.25), (4.29) and (4.30) that

$$\Phi(t w_n) \le \frac{t^2}{2} \left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2} \delta_n \right] - \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}} \left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}} \right) \right]^2 \left(\int_0^{d/n} r Q(r) \mathrm{d}r \right)^2$$

$$\le \frac{t^2}{2} \left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2} \delta_n \right]$$

$$- \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}} \left(\frac{t^2 \log n}{2\pi} \right)^{-1} \frac{\kappa^2}{4} e^{\pi^{-1}\beta_0 t^2 \log n} \left(\int_0^{d/n} r Q(r) \mathrm{d}r \right)^2$$

$$\le \frac{t^2}{2} \left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3 \log n} \right] - \frac{\pi^3 \kappa^2 \zeta_0^2 d^{2+\alpha+2b_0}}{2^{4-\alpha} n^{2+\alpha+2b_0} t^2 \log n} e^{\pi^{-1}\beta_0 t^2 \log n}$$

$$\le \left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3 \log n} \right] \frac{(2+\alpha+2b_0)\pi(1+\kappa)}{2\beta_0} - \frac{\pi^2 \kappa^2 \zeta_0^2 d^{2+\alpha+2b_0} \beta_0 n^{(2+\alpha+2b_0)\kappa}}{2^{4-\alpha}(2+\alpha+2b_0)(1+\kappa)\log n}$$

$$\le \frac{(2+\alpha+2b_0)\pi}{3\beta_0}, \tag{4.38}$$

which implies that there exists $\bar{n} \in \mathbb{N}$ such that (4.26) holds. In the above derivation process, we use the fact that the function

$$h_n(t) := \frac{t^2}{2} \left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3 \log n} \right] - \frac{\pi^3 \kappa^2 \zeta_0^2 d^{2+\alpha+2b_0}}{2^{4-\alpha} n^{2+\alpha+2b_0} t^2 \log n} e^{\pi^{-1}\beta_0 t^2 \log n}$$

is decreasing on $t \in \left(\sqrt{\frac{(2+\alpha+2b_0)\pi}{\beta_0}(1+\kappa)}, +\infty \right)$, since its stagnation points tend to $\sqrt{\frac{(2+\alpha+2b_0)\pi}{\beta_0}}$ as $n \to \infty$. $\square$

Lemma 4.5 *Assume that* (V1), (Q1), (Q2), (F1$'$), *and* (F2)-(F5) *hold. Then Φ satisfies* $(C)_c$ *condition.*

Proof Applying Lemmas 4.1 and 4.2, we deduce that there exists a sequence $\{u_n\} \subset E$ satisfying (4.1) and $\|u_n\| \le C_{10}$ for some constant $C_{10} > 0$. Since $\|u_n\| \le C_{10}$, by Lemma 2.4, we may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$

in $L^s(\mathbb{R}^2, Q)$ for $s \in [2, \infty)$ and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. It follows from (2.17) and (4.1) that

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n \le C_{11}. \tag{4.39}$$

In view of Lemma 4.3, for any $\phi \in E \cap \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi \mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi \mathrm{d}x. \tag{4.40}$$

By (4.1) and (4.40), it is easy to deduce that $\Phi'(\bar{u}) = 0$. Then by (F3), one has

$$\Phi(\bar{u}) = \Phi(\bar{u}) - \frac{1}{\mu}\langle\Phi'(\bar{u}), \bar{u}\rangle \ge 0.$$

By (4.39) and Lemma 4.3, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)F(\bar{u})\mathrm{d}x. \tag{4.41}$$

Thus

$$\begin{aligned}
c + o(1) = \Phi(u_n) &= \frac{1}{2}\|u_n - \bar{u}\|^2 + \frac{1}{2}\|\bar{u}\|^2 \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)F(\bar{u})\mathrm{d}x + o(1) \\
&= \frac{1}{2}\|u_n - u\|^2 + \Phi(\bar{u}) + o(1) \\
&\ge \frac{1}{2}\|u_n - u\|^2 + o(1).
\end{aligned}$$

Hence, it follows from (2.16), (4.1), (4.26) and (4.41) that

$$\|u_n - u\|^2 \le 2c + o(1) := \frac{(2 + \alpha + 2b_0)\pi}{\beta_0}(1 - 3\bar{\varepsilon}) + o(1). \tag{4.42}$$

Lemma 4.4 implies that $\bar{\varepsilon} > 0$. Now we choose $q \in (1, 2)$ such that

$$\frac{(1 + \bar{\varepsilon})^2(1 - 3\bar{\varepsilon})q^2}{1 - \bar{\varepsilon}} < 1. \tag{4.43}$$

By (2.16) and (4.1), we can deduce that

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n) \le C_{12}. \tag{4.44}$$

Let $q' = q/(q-1)$. Then $q' > 2$. Hence it follows from (4.42), (4.43), (4.44), (2.1), Lemma 2.5-ii), $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for $s \ge 2$, Hölder inequality and Young inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u})\right| \mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^2} \left|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n)\right| \mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^2} \left|[I_\alpha * (Q(|x|)f(u_n)(u_n - \bar{u}))]Q(|x|)f(u_n)(u_n - \bar{u})\right| \mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C_{13} \left(\int_{\mathbb{R}^2} \left| Q(|x|) f(u_n)(u_n - \bar{u}) \right|^{\frac{4}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4}}$$

$$\leq C_{14} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 dx \right)^{\frac{4\alpha}{(2+\alpha)^2}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^2 dx \right)^{\frac{8}{(2+\alpha)^2}}$$

$$+ C_{14} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{4q}{2+\alpha} \beta_0 (1+\bar{\varepsilon}) u_n^2} - 1 \right) dx \right)^{\frac{2+\alpha}{4q}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^{\frac{4q'}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4q'}}$$

$$\leq C_{14} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 dx \right)^{\frac{4\alpha}{(2+\alpha)^2}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^2 dx \right)^{\frac{8}{(2+\alpha)^2}}$$

$$+ C_{14} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{4q}{2+\alpha} \beta_0 (1+\bar{\varepsilon})^2 (u_n - \bar{u})^2} e^{\frac{4q}{2+\alpha} \beta_0 (1+\bar{\varepsilon})^2 \varepsilon^{-1} \bar{u}^2} - 1 \right) dx \right)^{\frac{2+\alpha}{4q}}$$

$$\times \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^{\frac{4q'}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4q'}}$$

$$\leq C_{14} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 dx \right)^{\frac{4\alpha}{(2+\alpha)^2}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^2 dx \right)^{\frac{8}{(2+\alpha)^2}}$$

$$+ C_{14} \left[\frac{1}{q} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{4q^2}{2+\alpha} \beta_0 (1+\bar{\varepsilon})^2 \|u_n - \bar{u}\|^2 [(u_n - \bar{u})/\|u_n - \bar{u}\|]^2} - 1 \right) dx \right.$$

$$\left. + \frac{1}{q'} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\frac{4qq'}{2+\alpha} \beta_0 (1+\bar{\varepsilon})^2 \varepsilon^{-1} \bar{u}^2} - 1 \right) dx \right]^{\frac{2+\alpha}{4q}}$$

$$\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^{\frac{4q'}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4q'}}$$

$$\leq C_{15} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^2 dx \right)^{\frac{8}{(2+\alpha)^2}} + C_{16} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n - \bar{u}|^{\frac{4q'}{2+\alpha}} dx \right)^{\frac{2+\alpha}{4q'}}$$

$$= o(1). \tag{4.45}$$

Which yields that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|) F(u_n))] Q(|x|) f(u_n)(u_n - \bar{u}) dx = 0. \tag{4.46}$$

Similarly, we can deduce that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|) F(\bar{u}))] Q(|x|) f(\bar{u})(u_n - \bar{u}) dx = 0. \tag{4.47}$$

By (4.1), it is obvious that

$$\langle \Phi'(u_n) - \Phi'(\bar{u}), u_n - \bar{u} \rangle = o(1). \tag{4.48}$$

From (4.46), (4.47) and (4.48), we can get

$$\|u_n - \bar{u}\|^2 = \langle \Phi'(u_n) - \Phi'(\bar{u}), u_n - \bar{u} \rangle - \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|) F(u_n))] Q(|x|) f(u_n)(u_n - \bar{u}) dx$$

$$+ \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|) F(\bar{u}))] Q(|x|) f(\bar{u})(u_n - \bar{u}) dx$$

$$= o(1).$$

$\square$

Lemma 4.6 *Assume that* (V1), (Q1), (F1$'$)*,* (F2) *and* (F3) *hold. Then*

i) $\sigma_0 := \inf\{\|u\| : u \in \mathcal{M}\} > 0$;
ii) $m := \inf\{\Phi(u) : u \in \mathcal{M}\} > 0$.

The proof is similar to Lemma 3.5, so we omit it here.

Proof of Theorem 1.4 First, we prove that $\mathcal{M} \neq \emptyset$. Applying Lemmas 4.1 and 4.2, we deduce that there exists a sequence $\{v_n\} \subset E$ satisfying (4.1) and $\|v_n\| \leq C_{17}$ for some constant $C_{17} > 0$. Hence, there exists $v \in E$ such that, up to a subsequence, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and $v_n \to v$ a.e. in $\mathbb{R}^2$. By Lemma 4.5, we know that $v_n \to v$ in E. Thus

$$\Phi(v) = \lim_{n\to\infty} \Phi(v_n) = c > 0, \quad \Phi'(v) = \lim_{n\to\infty} \Phi'(v_n) = 0.$$

Which imply that $v \neq 0$ and $v \in \mathcal{M}$.

Next, we prove that $m := \inf_{\mathcal{M}} \Phi(u)$ is achieved. By Lemma 4.6, we have $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. As in the proof of Lemma 4.2, we can deduce that $\{u_n\}$ is bounded in E. Hence, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and $u_n \to u$ a.e. in $\mathbb{R}^2$. In view of Lemma 4.3, for any $\phi \in E \cap \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))] Q(|x|) f(u_n) \varphi \, \mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))] Q(|x|) f(u) \varphi \, \mathrm{d}x. \tag{4.49}$$

By (4.1) and (4.49), it is easy to deduce that $\Phi'(u) = 0$. From the weak lower semicontinuous of norm and Fatou's lemma, we can deduce that

$$
\begin{aligned}
m &= \lim_{n\to\infty} \Phi(u_n) = \lim_{n\to\infty} \left(\Phi(u_n) - \frac{1}{\mu} \langle \Phi'(u_n), u_n \rangle \right) \\
&= \lim_{n\to\infty} \left(\frac{\mu - 1}{2\mu} \|u_n\|^2 + \frac{1}{2\mu} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))] Q(|x|) [f(u_n)u_n - \mu F(u_n)] \, \mathrm{d}x \right) \\
&\geq \frac{\mu - 1}{2\mu} \|u\|^2 + \frac{1}{2\mu} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))] Q(|x|) [f(u)u - \mu F(u)] \, \mathrm{d}x \\
&= \Phi(u) - \frac{1}{\mu} \langle \Phi'(u), u \rangle = \Phi(u) \geq m.
\end{aligned}
$$

This shows that $u \in \mathcal{M}$ and $\Phi(u) = m$. $\square$

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