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Superlinear singular double phase problems

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Abstract. We examine three singular Dirichlet problems driven by the double phase operator. One of the problems is nonparametric and the other two are parametric. In all problems, the perturbation is "superlinear", but does not satisfy the Ambrosetti-Rabinowitz condition. We prove existence and multiplicity results for the problems. For the parametric problems, the results are global in the parameter $\lambda > 0$. Our approach uses variational tools from the critical point theory, truncations and comparisons and critical groups.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N (N \geq 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following singular Dirichlet problems:

$$\left\{\begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \beta(z)u(z)^{-\eta} + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0,\ 0 < \eta < 1 < q < p < N. \end{array}\right\} \text{(nonparametric problem)} \tag{P}$$

$$\left\{\begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda\beta(z)u(z)^{-\eta} + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0,\ 0 < \eta < 1 < q < p < N,\ \lambda > 0. \end{array}\right\} \text{(parametric problem 1)} \tag{$(P_\lambda)^1$}$$

$$\left\{\begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \beta(z)u(z)^{-\eta} + \lambda f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0,\ 0 < \eta < 1 < q < p < N,\ \lambda > 0. \end{array}\right\} \text{(parametric problem 2)} \tag{$(P_\lambda)^2$}$$

Given $a \in C^{0,1}(\bar{\Omega})\backslash\{0\}$ with $a(z) \geq 0$ for all $z \in \bar{\Omega}$, by Δ_p^a we denote the weighted p-Laplace differential operator (with weight $a(\cdot)$) defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

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If $\min_{\bar{\Omega}} a = c_0 > 0$(that is, the weight $a(\cdot)$ is bounded away from zero), then Δ_p^a is just a nonautonomous version of the standard p-Laplacian and so it can be treated using the same tools. The differential operator driving our problems, is associated with the density function $\vartheta(z,t)$ defined by

$$\vartheta(z,t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

When $\min_{\bar{\Omega}} a = c_0 > 0$, the function $\vartheta(z,\cdot)$ exhibits balanced growth, namely we have

$$c_0 t^p \leq \vartheta(z,t) \leq c_1[1+t^p] \quad \text{for all } z \in \Omega, \text{ all } t \geq 0, \text{some } c_1 > 0.$$

So, $\vartheta(z,\cdot)$ is controlled from above and below by same powers of $t \geq 0$. This property leads to a global (that is, up to the boundary $\partial\Omega$) regularity theory of Lieberman [19]. This theory provides many powerful analytical tools for the study of the problems. We mention some characteristic such tools:

- The equivalence of Sobolev and Hölder local minimizers of the energy functional (see Brezis-Nirenberg [4] and García Azorero-Peral Alonso-Manfredi [8]).
- Nonlinear versions of the Hopf maximum principle (see Pucci-Serrin [33]).
- Strong comparison results (see Papageorgiou-Rădulescu-Repovš [29]).
- Unbalanced growth problems, are studied using the classical Lebesgue and Sobolev spaces.

The situation changes if $\min_{\bar{\Omega}} a = 0$. Then the density function $\vartheta(z,\cdot)$ exhibits unbalanced growth, namely we have

$$t^q \leq \vartheta(z,t) \leq c_1[1+t^p] \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

We see that in this case, the growth of $\vartheta(z,\cdot)$ is trapped between two distinct powers of $t \geq 0$. Now the global regularity theory of Lieberman [19] is not valid and so the analytical tools mentioned above, are not available. In fact, for problems with unbalanced growth (also known as double phase problems or problems with (p,q)-growth), there is no global regularity theory, only local regularity results. We refer to the survey papers of Marcellini [25] and Mingione-Rădulescu [26]. Moreover, for these problems, the classical Lebesgue and Sobolev spaces, do not provide an adequate functional framework for the analysis of the equation. We need to pass to the broader class of the generalized Orlicz spaces (see Harjulehto-Hästö [15] and Papageorgiou-Winkert [32]).

In problems $(P_\lambda), (P_\lambda)^1, (P_\lambda)^2$, we encounter one more difficulty, which is the presence of the singular term. This leads to an energy functional which is not C^1. So, we can not use the minimax theorems of the critical point theory. We have to find ways to bypass the singularity and deal with C^1-functionals. This is not an easy task, due to the lack of a global regularity theory. In our problems, the singular term is perturbed by a function $f(z,x)$, which is assumed to be $(p-1)$-superlinear as $x \to +\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). In problems (P) and $(P_\lambda)^1$, we do not assume that $f \geq 0$. So, the perturbation can be sign-changing.

The study of singular double phase problems, is lagging behind. There are few works in the literature on the subject. We mention the papers of Arora-Fiscella-Mukherjee-Winkert [2], Bai-Papageorgiou-Zeng [3], Liu-Dai-Papageorgiou-Winkert [20], Liu-Papageorgiou [21], Liu-Winkert [22]. With the exception of [3], the other works assume that the perturbation is of power-type (that is, $f(z, u) = u^{r-1}$ with $p < r$) and use the Nehari method to produce two solutions. In [3], there are two parameters in the problem and the perturbation $f(z, x)$ is nonnegative (the reaction has the form $u \mapsto \mu u^{-\eta} + \lambda f(z, u)$ with $\mu, \lambda > 0$, $f \geq 0$). Very recently, Han-Liu-Papageorgiou [14], studied singular double phase problems, with a $(p-1)$-superlinear perturbation $f(z, x)$ which is sign changing. Their result does not cover the case of a nonnegative perturbation, since they require that $f(z, \cdot)$ is nonpositive near zero (see hypothesis H_1(iii) in [14]). Here, for problem $(P_\lambda)^1$, we provide a unified treatment of both the definite and indefinite perturbation, extending the work of [14]. Finally, we also mention the very recent work of Guarnotta-Winkert [11], where the authors study a nonlocal version of problem $(P_\lambda)^1$ (Kirchhoff problem) and using the Nehari method, they prove the existence of two solutions with opposite energy signs, where $\lambda > 0$ is small. There result is not global in $\lambda > 0$.

We mention that boundary value problems and variational integrals with unbalanced (or nonstandard) growth, are used to describe anisotropic materials. The modulating coefficient (weight) $a(\cdot)$, dictates the structure of composites made of two different materials with distinct hardening exponents p (when $a(z) > 0$) and q (when $a(z) = 0$), that is, the material has two phases. Other physical applications and applications to calculus of variations (Lavrentiev phenomenon), can be found in the paper of Zhikov [34]. From a purely mathematical viewpoint, in a double phase problem the differential operator $u \mapsto \Delta_p^a u + \Delta_q u$ has nonuniform ellipticity in Ω.

2. Mathematical background

Let $L^0(\Omega)$ be the space of measurable functions $u : \Omega \mapsto \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue null set. Also, let $a \in C^{0,1}(\bar{\Omega}) \backslash \{0\}, a(z) \geq 0$. Recall that $\vartheta(z, t)$ is the density function defined by

$$\vartheta(z, t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

The Lebesgue-Orlicz space $L^\vartheta(\Omega)$, is defined by

$$L^\vartheta(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\vartheta(u) = \int_\Omega \vartheta(z, |u|)\, dz < \infty \right\}.$$

We call $\rho_\vartheta(\cdot)$ the "modular function" corresponding to the density $\vartheta(z, x)$. We endow $L^\vartheta(\Omega)$ with the so-called "Luxemburg norm" $\|\cdot\|_\vartheta$ defined by

$$\|u\|_\vartheta = \inf \left\{ \lambda > 0 : \rho_\vartheta \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

With this norm $L^\vartheta(\Omega)$ becomes a Banach space which is separable and reflexive. In fact, it is uniformly convex, since the function $t \mapsto \vartheta(z, t)$ is uniformly convex. Recall that a uniformly convex space, is reflexive (Milman-Pettis theorem, see Papageorgiou-Winkert [32], p.243).

Using $L^{\vartheta}(\Omega)$, we can define the corresponding Sobolev-Orlicz space $W^{1,\vartheta}(\Omega)$ by

$$W^{1,\vartheta}(\Omega) = \left\{u \in L^{\vartheta}(\Omega) : |Du| \in L^{\vartheta}(\Omega)\right\},$$

with Du being the weak gradient of u. We endow $W^{1,\vartheta}(\Omega)$ with the natural norm $\|\cdot\|_{1,\vartheta}$, defined by

$$\|u\|_{1,\vartheta} = \|u\|_{\vartheta} + \|Du\|_{\vartheta} \quad \text{for all } u \in W^{1,\vartheta}(\Omega),$$

where $\|Du\|_{\vartheta} = \||Du|\|_{\vartheta}$. Also, if $C_c^{\infty}(\Omega)$ denotes the space of C^{∞}-functions with compact support, then we define

$$W_0^{1,\vartheta}(\Omega) = \overline{C_c^{\infty}(\Omega)}^{\|\cdot\|_{1,\vartheta}}.$$

Both $W^{1,\vartheta}(\Omega)$ and $W_0^{1,\vartheta}(\Omega)$ are Banach spaces, which are separable and reflexive (in fact uniformly convex). Moreover, on $W_0^{1,\vartheta}(\Omega)$ the Poincaré inequality is true, namely we can find $c_2 > 0$ such that

$$\|u\|_{\vartheta} \leq c_2 \|Du\|_{\vartheta} \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega).$$

Therefore, on $W_0^{1,\vartheta}(\Omega)$, we can use the equivalent norm $\|\cdot\|$ defined by

$$\|u\| = \|Du\|_{\vartheta} \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega). \tag{1}$$

There are some useful embeddings between these spaces. Recall that $q^* = \frac{Nq}{N-q}$. The symbol $\hookrightarrow$ denotes continuous and dense embedding.

Proposition 2.1. *The following results hold:*

(a) $L^{\vartheta}(\Omega) \hookrightarrow L^q(\Omega)$ *and* $W_0^{1,\vartheta}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$.
(b) $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $1 \leq r \leq q^*$ *and the embedding is compact if* $1 \leq r < q^*$.
(c) $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,\vartheta}(\Omega)$.

Also there is a close relation between the modular function $\rho_{\vartheta}(\cdot)$ and the norm $\|\cdot\|$ (see (1)).

Proposition 2.2. *The following results hold:*

(a) *If* $u \neq 0$, *then* $\|u\| = \mu \Leftrightarrow \rho_{\vartheta}\left(\frac{Du}{\mu}\right) = 1$.
(b) $\rho_{\vartheta}(Du) < 1$(*resp.* $= 1, > 1$)$\Leftrightarrow \|u\| < 1$(*resp.* $= 1, > 1$).
(c) *If* $\|u\| < 1$, *then* $\|u\|^p \leq \rho_{\vartheta}(Du) \leq \|u\|^q$.
(d) *If* $\|u\| > 1$, *then* $\|u\|^q \leq \rho_{\vartheta}(Du) \leq \|u\|^p$.
(e) $\|u\| \to 0$*(resp.* $\to +\infty$*)*$\Leftrightarrow \rho_{\vartheta}(Du) \to 0$(*resp.* $\to +\infty$).

Let $V : W_0^{1,\vartheta}(\Omega) \mapsto W_0^{1,\vartheta}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_{\Omega} \left[a(z)|Du|^{p-2} + |Du|^{q-2}\right] (Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,\vartheta}(\Omega).$$

This operator has the following properties (see Papageorgiou-Winkert [32], p.683).

Proposition 2.3. *$V(\cdot)$ is bounded (that is maps bounded sets to bounded ones), continuous, strictly monotone, coercive and of type $(S)_+$, that is,*

"$u_n \xrightarrow{w} u$ in $W_0^{1,\vartheta}(\Omega)$, $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leq 0$ imply that $u_n \to u$ in $W_0^{1,\vartheta}(\Omega)$".

Remark 2.4. $V(\cdot)$ being continuous, strictly monotone, it is maximal monotone (see [32], p.565). Recall that a maximal monotone, coercive operator, is surjective (see [32], p.576). Therefore $V(\cdot)$ is surjective (in fact it is a homeomorphism).

In the study of singular problems, useful is the so called "Hardy's inequality"(see, for example, Papageorgiou-Rădulescu-Repovš [28], p.66). Let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \Omega$.

Proposition 2.5. *There exists $c_3 > 0$ such that*

$$\left\| \frac{u}{\hat{d}} \right\|_{\mu} \leq c_3 \|Du\|_{\mu}$$

for all $u \in W_0^{1,\mu}(\Omega), 1 < \mu < \infty$.

The constant $c_3 > 0$ depends on (Ω, p) and it is bounded in $p > 1$. Indeed, from Hajlasz [13, Proposition 1], we know that

$$\frac{u(z)}{\hat{d}(z)} \leq C|M(Du)|(z) \text{ for all } z \in \Omega,$$

with $M(\cdot)$ being the Hardy-Littlewood maximal map, where $C = C(N) > 0$. Then by Theorem 4.3.8 of Diening-Harjulehto-Hästö-Ružička [7, p.110], we have that

$$\left\| \frac{u}{\hat{d}} \right\|_{p} \leq K \frac{p}{p-1} \|Du\|_p \text{ with } K = K(N) > 0.$$

Given $u \in L^0(\Omega)$, we define

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}.$$

Then $u = u^+ - u^-$ and $|u| = u^+ + u^-$. Moreover, if $u \in W_0^{1,\vartheta}(\Omega)$, then $u^+, u^- \in W_0^{1,\vartheta}(\Omega)$. Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Consider the problem

$$-\Delta_p^a u(z) - \Delta_q u(z) = g(z, u(z)) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2}$$

We assume that $g : \Omega \times \mathring{\mathbb{R}}_+ \mapsto \mathbb{R}$ ($\mathring{\mathbb{R}}_+ = (0, \infty)$) is a Carathéodory function (that is, $z \mapsto g(z, x)$ is measurable and $x \mapsto g(z, x)$ is continuous) which satisfies

$$|g(z, x)| \leq c_4[1 + x^{-\eta} + x^{r-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x > 0, \text{ some } c_4 > 0.$$

By a solution (weak solution) of (2), we mean a function $u \in W_0^{1,\vartheta}(\Omega)$ such that

$$u(z) > 0 \quad \text{for a.a. } z \in \Omega,\ g(\cdot, u(\cdot))h \in L^1(\Omega) \text{ for all } h \in W_0^{1,\vartheta}(\Omega),$$

$$\langle V(u), h\rangle = \int_\Omega g(z,u)h\,dz \quad \text{for every } h \in W_0^{1,\vartheta}(\Omega).$$

Proposition 2.6. *If $a \in C^{0,1}(\bar{\Omega})\backslash\{0\}$, $a(z) \geq 0$ for all $z \in \bar{\Omega}$, $0 < \eta < 1 < q < p < r < q^*$ and $u \in W_0^{1,\vartheta}(\Omega)$ is a solution of (2), then $u \in L^\infty(\Omega)$.*

Proof. For $k > 1$, we introduce the set

$$E_k = \{z \in \Omega : u(z) > k\}.$$

Choose $k^* > 1$ large such that

$$\|(u-k)^+\| \leq 1 \quad \text{for all } k \geq k^*. \tag{3}$$

We have

$$\langle V(u), h\rangle = \int_\Omega g(z,u)h\,dz \quad \text{for all } h \in W_0^{1,\vartheta}(\Omega).$$

We choose the test function $h = (u-k)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned} \|(u-k)^+\|^p &\leq \rho_\vartheta(D(u-k)^+)(\text{see (3) and Proposition (2.2)}) \\ &= \int_\Omega g(z,u)(u-k)^+\,dz \\ &\leq c_4 \int_\Omega [1 + u^{-\eta} + u^{r-1}](u-k)^+\,dz. \end{aligned} \tag{4}$$

On the set $E_k (k \geq k^* > 1)$, we have

$$(u-k)^{1-\eta} \leq u^{1-\eta} \leq u \leq u^r, \ (u-k)^r \leq u^r. \tag{5}$$

From (4), (5), we obtain

$$\begin{aligned} \|(u-k)^+\|^p &\leq c_5 \int_\Omega \chi_{E_k} u^r\,dz \quad \text{for some } c_5 > 0 \\ &= c_5 \int_\Omega \chi_{E_k}[(u-k)^+]^r\,dz. \end{aligned} \tag{6}$$

By hypothesis $p < r < q^*$. Then

$$\chi_{E_k} \in L^{\frac{q^*}{q^*-r}}(\Omega), \quad [(u-k)^+]^r \in L^{\frac{q^*}{r}}(\Omega)(\text{note that } W_0^{1,\vartheta}(\Omega) \hookrightarrow L^{q^*}(\Omega), \text{see Proposition 2.1}).$$

From (6), using Hölder's inequality, we obtain

$$\|(u-k)^+\|^p \leq c_6 |E_k|_N^{\frac{q^*-r}{q^*}} \|(u-k)^+\|^r \quad \text{for some } c_6 > 0$$
$$(\text{note that } W_0^{1,\vartheta}(\Omega) \hookrightarrow L^{\frac{q^*}{r}}(\Omega), \text{see Proposition 2.1}).$$

Let $1 < \mu < p < r$. On account of (3), we have

$$\|(u-k)^+\|^p \leq c_6 |E_k|_N^{\frac{q^*-r}{q^*}} \|(u-k)^+\|^\mu,$$
$$\Rightarrow \|(u-k)^+\|^{p-\mu} \leq c_6 |E_k|_N^{\frac{q^*-r}{q^*}}. \tag{7}$$

Let $m > k$. Then

$$
\begin{aligned}
(m-k)^r |E_m|_N^{\frac{r}{q^*}} &\le \left[\int_{E_m} |u-k|^{q^*}\,dz\right]^{\frac{r}{q^*}} \\
&\le \left[\int_{E_k} |u-k|^{q^*}\right]^{\frac{r}{q^*}} \text{ (since } E_m \subseteq E_k) \\
&\le c_7 \|(u-k)^+\|^r \quad \text{for some } c_7 > 0 \\
&\quad \text{(use that } W_0^{1,\vartheta}(\Omega) \hookrightarrow L^{q^*}(\Omega), \text{see Proposition 2.1)},
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (m-k)^{p-\mu} |E_m|_N^{\frac{p-\mu}{q^*}} &\le c_8 \|(u-k)^+\|^{p-\mu} \quad \text{for some } c_8 > 0 \\
&\le c_9 |E_k|_N^{\frac{q^*-r}{q^*}} \quad \text{with } c_9 = c_8 \cdot c_6 > 0 \text{ (see (7))},
\end{aligned}
$$

$$
\Rightarrow |E_m|_N \le \frac{c_{10}}{(m-k)^{q^*}} |E_k|_N^{\frac{q^*-r}{p-\mu}} \quad \text{for some } c_{10} > 0. \tag{8}
$$

We have

$$
0 < \epsilon = q^* - r \text{ (recall } r < q^*).
$$

So, we choose $\mu < p$ such that

$$
0 < \frac{\epsilon}{2} = p - \mu.
$$

Then $1 < \frac{q^*-r}{p-\mu}$. So, from (8) and Lemma B.1, p.63, of Kinderlehrer-Stampacchia [18](with $\varphi(t) = |E_t|_N, t \ge k^*$) we find $M \ge k^*$ large such that

$$
|E_M|_N = 0, \Rightarrow u \in L^\infty(\Omega).
$$

This completes the proof. □

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following two sets

$$
\begin{aligned}
&K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (critical set of } \varphi(\cdot)), \\
&\varphi^c = \{u \in X : \varphi(u) \le c\}.
\end{aligned}
$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

"Every $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

This is a compactness-type condition on the functional $\varphi(\cdot)$, to compensate for the fact that X is not locally compact, being in general infinite dimensional.

Let $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $H_k(Y_1, Y_2)$ we denote the k-th singular homology group with coefficients in $\mathbb{R}$, for the pair (Y_1, Y_2). Since $\mathbb{R}$ is a field, each $H_k(Y_1, Y_2)$ is a linear space(we use $\mathbb{R}$ in order to avoid torsion phenomena). Let $u \in K_\varphi$ be isolated and $c = \varphi(u)$. Then the critical groups of $\varphi(\cdot)$ at u, are defined by

$$
C_k(\varphi, u) = H_k(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \setminus \{u\}),\ k \in \mathbb{N}_0,
$$

where $\mathcal{U}$ is a neighborhood of u such that

$$K_\varphi \cap \varphi^c \cap \mathcal{U} = \{u\}.$$

The excision property of singular homology (see [28], Section 6.1), implies that this definition is independent of the choice of the isolating neighborhood $\mathcal{U}$.

If $m, n \in \mathbb{N}_0$, then by $\delta_{m,n}$ we denote the Kronecker symbol defined by

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

A function $\gamma : \mathbb{R} \mapsto \mathbb{R}$ is said to be locally Lipschitz if for every $K \subseteq \mathbb{R}$ compact $\gamma|_K$ is Lipschitz continuous with Lipschitz constant $c_K > 0$. A function $\gamma : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is said to be $L^\infty(\Omega)$-locally Lipschitz if $z \mapsto \gamma(z, x)$ is measurable and $x \mapsto \gamma(z, x)$ is locally Lipschitz with Lipschitz constant $c_K \in L^\infty(\Omega)$ for every $K \subseteq \mathbb{R}$ compact. Note that such a function $\gamma(z, x)$ is jointly measurable(see Papageorgiou-Winkert [32], p.108).

If $u \in L^0(\Omega)$, then we write that $0 \prec u$, if for every $D \subseteq \Omega$ compact, we have

$$0 < c_D \leq u(z) \quad \text{for a.a. } z \in D.$$

Let $C_0^1(\bar{\Omega}) = \left\{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\right\}$. This is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Also, in what follows $\rho_a : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ is the modular function defined by

$$\rho_a(Du) = \int_\Omega a(z)|Du|^p \, dz \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega).$$

Finally by $\hat{\lambda}_1(q)$ we denote the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. We know that $\hat{\lambda}_1(q) > 0$ and

$$\hat{\lambda}_1(q) = \inf\left\{\frac{\|Du\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}(\Omega), u \neq 0\right\}.$$

3. Problem (P)

In the literature, there are no multiplicity results for nonparametric singular double phase problems. The presence of a parameter $\lambda > 0$(multiplying either the singular term or the perturbation), is very helpful because by varying $\lambda > 0$, we can achieve geometric configurations which permit the use of the minimax theorems from the critical point theory.

For $\epsilon > 0$ small, $\Omega_\epsilon = \{z \in \Omega : d(z, \partial\Omega) < \epsilon\}$. The hypotheses on the data of (P) are following:

(H_0) $a \in C^{0,1}(\bar{\Omega})\setminus\{0\}, a(z) \geq 0$ for all $z \in \bar{\Omega}$, $0 < \eta < 1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$ and $\beta \in W_0^{1,\infty}(\Omega)$ with $\beta(z) > 0$ for all $z \in \Omega$ and there exist $\epsilon_0 > 0$ and $c^* > 0$ such that $c^* \hat{d}^\eta \leq \beta$ in Ω_{ϵ_0}.

Remark 3.1. The condition $\frac{p}{q} < 1 + \frac{1}{N}$ says that the two exponents can not be for apart. It is common in double phase problems and leads to local regularity results which in turn are used to prove a weak maximum principle (see Papageorgiou-Vetro-Vetro [31]). We know that every element in $W_0^{1,\infty}(\Omega)$ has a representative which is Lipschitz continuous. Therefore $0 \prec \beta$.

(H_1): $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is an $L^\infty(\Omega)$-locally Lipschitz function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq \hat{a}(z)[1 + x^{r-1}]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\hat{a} \in L^\infty(\Omega), p < r < q^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)\, ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exist $\hat{c} > 0$ and $\tau \in [1, q)$ such that

$$-\hat{c}[1 + x^\tau] \leq f(z, x)x - pF(z, x) \text{ for a.a. } z \in \Omega,\ \text{all } x \geq 0;$$

(iii) there exist $0 < \delta < \gamma_0$ such that

$$0 \leq f(z, x) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leq x \leq \delta,\ \beta(z)\gamma_0^{-\eta} + f(z, \gamma_0) \leq 0 \text{ for a.a. } z \in \Omega.$$

Remark 3.2. The absence of a parameter in the equation, forces us to assume that $f(z, \cdot)$ exhibits an oscillatory behavior near zero (see hypothesis H_1(iii)). Hypothesis H_1(ii) implies that

$$\lim_{|x|\to+\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

that is, the perturbation $f(z, \cdot)$ is $(p - 1)$-superlinear. However, we do not use the Ambrosetti-Rabinowitz condition(see Ambrosetti-Rabinowitz [1]) or any of its generalizations existing in the literature (see, for example, Li-Yang [24]). Since we look for positive solutions, the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$ and $f(z, 0) = 0$, we may assume without any loss of generality that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$.

We start by considering the following purely singular problem

$$-\Delta_p^a u(z) - \Delta_q u(z) = \lambda\beta(z)u(z)^{-\eta} \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ \lambda > 0. \tag{9}$$

For this problem, we have the following result.

Proposition 3.3. *If hypotheses H_0 hold and $\lambda > 0$, then problem (9) has a unique solution*

$$\bar{u}_\lambda \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec \bar{u}_\lambda,$$
$$\{u_\lambda\}_{\lambda>0} \textit{ is nondecreasing},$$
$$\bar{u}_\lambda \to 0 \textit{ in } W_0^{1,\vartheta}(\Omega) \textit{ as } \lambda \to 0^+.$$

Proof. From Proposition 3.1 of Bai-Papageorgiou-Zeng [3], we know that problem (1) has a unique solution

$$\bar{u}_\lambda \in W_0^{1,\vartheta}(\Omega),\ 0 \prec \bar{u}_\lambda.$$

Proposition 2.6, implies that

$$\bar{u}_\lambda \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega).$$

Moreover, note that $-\Delta_p^a \bar{u}_\lambda - \Delta_q \bar{u}_\lambda \geq 0$ in Ω and so Proposition 2.4 of Papageorgiou-Vetro-Vetro [31], implies that $0 \prec \bar{u}_\lambda$.

Let $0 < \mu < \lambda$. We have

$$-\Delta_p^a \bar{u}_\lambda - \Delta_q \bar{u}_\lambda \geq \mu\beta(z)\bar{u}_\lambda^{-\eta} \text{ in } \Omega. \tag{10}$$

We introduce the function $w_\mu(z,x)$ defined by

$$w_\mu(z,x) = \begin{cases} \mu\beta(z)(x^+)^{-\eta} & \text{if } 0 < x \leq \bar{u}_\lambda(z), \\ \mu\beta(z)\bar{u}_\lambda(z)^{-\eta} & \text{if } \bar{u}_\lambda(z) < x, \end{cases} \tag{11}$$

and consider the singular problem

$$-\Delta_p^a u(z) - \Delta_q u(z) = w_\mu(z,u(z)) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{12}$$

Reasoning as in Proposition 3.1 of Bai-Papageorgiou-Zeng [3], we show that problem (12) has a solution $\tilde{u}_\mu \in W_0^{1,\vartheta}(\Omega)$. We have

$$\langle V(\tilde{u}_\mu), h\rangle = \int_\Omega w_\mu(z,\tilde{u}_\mu)h\,dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega).$$

Choosing the test function $h = (\tilde{u}_\mu - \bar{u}_\lambda)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned} \langle V(\tilde{u}_\mu), (\tilde{u}_\mu - \bar{u}_\lambda)^+\rangle &= \int_\Omega \mu\beta(z)\bar{u}_\lambda^{-\eta}(\tilde{u}_\mu - \bar{u}_\lambda)^+\,dz \text{ (see (11))} \\ &\leq \langle V(\bar{u}_\lambda), (\tilde{u}_\mu - \bar{u}_\lambda)^+\rangle \text{ (see (10))}, \\ \Rightarrow\ & \tilde{u}_\mu \leq \bar{u}_\lambda \text{ (see Proposition 3)}, \\ \Rightarrow\ & \tilde{u}_\mu \text{ is a solution of (9) for the parameter } \mu. \end{aligned}$$

Since the solution of (9) is unique, we infer that

$$\begin{aligned} &\tilde{u}_\mu = \bar{u}_\mu, \\ \Rightarrow\ &\bar{u}_\mu \leq \bar{u}_\lambda, \\ \Rightarrow\ &\{\bar{u}_\lambda\}_{\lambda>0} \text{ is nondecreasing.} \end{aligned}$$

Finally, we know that

$$\langle V(\bar{u}_\lambda), h\rangle = \lambda \int_\Omega \beta(z)\bar{u}_\lambda^{-\eta}h\,dz \quad \text{for all } h \in W_0^{1,\vartheta}(\Omega).$$

Choosing the test function $h = \bar{u}_\lambda \in W_0^{1,\vartheta}(\Omega)$, we obtain

$\rho_\vartheta(\bar{u}_\lambda) \leq \lambda c_{11}\|\bar{u}_\lambda\|^{1-\eta}$ for some $c_{11} > 0$

(see Theorem 13.17 of Hewitt-Stromberg [16] and Proposition 2.1)

$\Rightarrow \bar{u}_\lambda \to 0$ in $W_0^{1,\vartheta}(\Omega)$ as $\lambda \to 0^+$.

The proof is now complete. □

Proposition 3.4. *If hypotheses H_0 hold and $\lambda > 0$, then $\beta(\cdot)\bar{u}_\lambda(\cdot)^{-\eta} \in L^\infty(\Omega)$.*

Proof. From Lemma 14.16, p.355, of Gilbarg-Trudinger [9], we can find $0 < \hat{\epsilon}_0 \leq \epsilon_0$ such that $\hat{d} \in C^2(\bar{\Omega}_{\hat{\epsilon}_0})$ and $|\nabla \hat{d}(z)| = 1$ for all $z \in \bar{\Omega}_{\epsilon_0}$. Let $\delta \in (0, \frac{\hat{\epsilon}_0}{2})$ and consider the function $\hat{k}(z)$ defined by

$$\hat{k}(z) = \begin{cases} \hat{d}(z) & \text{if } \hat{d}(z) < \delta, \\ \delta + \int_\delta^{\hat{d}(z)} \left[\frac{2\delta - s}{\delta}\right]^{\frac{1}{q-1}} ds & \text{if } \delta \leq \hat{d}(z) \leq 2\delta, \\ \delta + \int_\delta^{2\delta} \left[\frac{2\delta - s}{\delta}\right]^{\frac{1}{q-1}} ds & \text{if } 2\delta < \hat{d}(z). \end{cases}$$

Evidently $\hat{k} \in \operatorname{int} C_+$ and we have

$$\Delta_p^a \hat{k} + \Delta_q \hat{k} = \begin{cases} \Delta^{a+1}\hat{d} & \text{if } \hat{d}z < \delta, \\ \Delta^{al_1 + l_2}\hat{d} & \text{if } \delta \leq \hat{d}(z) \leq 2\delta, \\ 0 & \text{if } 2\delta < \hat{d}(z), \end{cases}$$

with $l_1(z) = \left[\frac{2\delta - \hat{d}(z)}{\delta}\right]^{\frac{p-1}{q-1}}$, $l_2(z) = \frac{2\delta - \hat{d}(z)}{\delta}$. It follows that

$$-\Delta_p^a \hat{k} - \Delta_q \hat{k} \in L^\infty(\Omega_{\hat{\epsilon}_0}). \tag{13}$$

Since $\hat{k} \in \operatorname{int} C_+$, using Lemma 2.3 of Guo-Webb [12], we can find $0 < c_{12} \leq c_{13}$ such that

$$c_{12}\hat{d} \leq \hat{k} \leq c_{13}\hat{d} \text{ in } \bar{\Omega}_{\hat{\epsilon}_0}. \tag{14}$$

For $t \in (0, 1)$, let $\hat{k}_t = t\hat{k}$. On account of (13), if $t \in (0, 1)$ is small, we have

$$\begin{aligned} -\Delta_p^a \hat{k}_t - \Delta_q k_t &\leq \frac{\lambda c^*}{c_{13}^\eta} \text{ (see hypotheses H}_0 \text{ and (14))} \\ &\leq \lambda \frac{\beta(z)}{(c_{13}\hat{d})^\eta} \text{ (see hypotheses H}_0 \text{ and recall } 0 < \hat{\epsilon}_0 \leq \epsilon_0) \\ &\leq \lambda\beta(z)\hat{k}(z)^{-\eta} \text{ (see (14))} \\ &\leq \lambda\beta(z)\hat{k}_t(z)^{-\eta} \text{ in } \Omega_{\hat{\epsilon}_0} \text{ (since } t \in (0, 1)). \end{aligned} \tag{15}$$

Moreover, since $0 \prec \bar{u}_\lambda$, we see that for $t \in (0, 1)$ small, we can have $\hat{k}_t|_{\partial\Omega_{\hat{\epsilon}_0}} \leq \bar{u}_\lambda|_{\partial\Omega_{\hat{\epsilon}_0}}$ and so from (15) and the weak comparison principle of Pucci-Serrin [33](p.61), we infer that

$$\begin{aligned} &k_t \leq \bar{u}_\lambda \text{ in } \Omega_{\hat{\epsilon}_0}, \\ &\Rightarrow c_{14}\hat{d} \leq \bar{u}_\lambda \text{ in } \bar{\Omega}_{\hat{\epsilon}_0}, \text{ for some } c_{14} > 0. \end{aligned} \tag{16}$$

Recall that $0 \prec \bar{u}_\lambda$. Therefore, if

$$L^\infty(\Omega \backslash \bar{\Omega}_{\hat{\epsilon}_0})_+ = \left\{u \in L^\infty(\Omega \backslash \bar{\Omega}_{\hat{\epsilon}_0}) : u(z) \geq 0 \text{ for a.a. } z \in \Omega \backslash \bar{\Omega}_{\hat{\epsilon}_0}\right\},$$

we know that

$$\emptyset \neq \operatorname{int} L^\infty(\Omega \backslash \bar{\Omega}_{\hat{\epsilon}_0})_+ = \left\{u \in L^\infty(\Omega \backslash \bar{\Omega}_{\hat{\epsilon}_0}) : \operatorname{essinf}_{\Omega \backslash \bar{\Omega}_{\hat{\epsilon}_0}} u > 0\right\}$$

and so we infer that

$$\bar{u}_\lambda \in \operatorname{int} L^\infty(\Omega \backslash \bar{\Omega}_{\epsilon_0})_+.$$

Then using Proposition 4.1.22, p.274, of Papageorgiou-Rădulescu-Repovš [28], we can find $c_{15} > 0$ such that

$$c_{15}\hat{d} \leq \bar{u}_\lambda \text{ in } \Omega\backslash\bar{\Omega}_{\epsilon_0}. \tag{17}$$

Combining (16), (17), we obtain

$$c_{16}\hat{d} \leq \bar{u}_\lambda \text{ in } \Omega \text{ with } c_{16} = \min\{c_{14}, c_{15}\} > 0. \tag{18}$$

Let $\mu > 1$. We have

$$\begin{aligned}
\int_\Omega \left(\frac{\beta(z)}{\bar{u}_\lambda^\eta}\right)^\mu dz &= \int_\Omega \left[\beta(z)\bar{u}_\lambda^{1-\eta}\frac{1}{\bar{u}_\lambda}\right]^\mu dz \\
&\leq c_{17}\int_\Omega \left(\frac{\beta(z)}{\hat{d}}\right)^\mu dz \text{ for some } c_{17} > 0 \\
&\quad \text{(see (18) and recall } \bar{u}_\lambda \in L^\infty(\Omega)\text{)}, \\
\Rightarrow \|\beta(\cdot)\bar{u}_\lambda(\cdot)^{-\eta}\|_\mu^\mu &\leq c_{18}\|D\beta\|_\mu^\mu \ \text{ with } c_{18} = \left(K\frac{p}{p-1}\right)^\mu \\
&\quad \text{(see Proposition (2.5))}, \\
\Rightarrow \|\beta(\cdot)\bar{u}_\lambda(\cdot)^{-\eta}\|_\mu &\leq K\frac{p}{p-1}\|D\beta\|_\mu.
\end{aligned}$$

Let $\mu \to \infty$ to conclude that

$$\|\beta(\cdot)\bar{u}_\lambda(\cdot)^{-\eta}\|_\infty \leq \|D\beta\|_\infty, \tag{19}$$

$$\Rightarrow \beta(\cdot)\bar{u}_\lambda(\cdot)^{-\eta} \in L^\infty(\Omega).$$

This completes the proof. □

From Propositions (3.3) and (3.4), we obtain the following result.

Proposition 3.5. *If hypotheses H_0 hold, then $\bar{u}_\lambda \to 0$ in $W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ as $\lambda \to 0^+$.*

Proof. We already know that

$$\bar{u}_\lambda \to 0 \text{ in } W_0^{1,\vartheta}(\Omega) \text{ as } \lambda \to 0^+ \text{ (see Proposition 3.3)}.$$

Also, from Proposition 7 of Papageorgiou-Rădulescu [27] we have

$$\begin{aligned}
\|\bar{u}_\lambda\|_\infty &\leq \lambda c_{19}\|\beta(\cdot)\bar{u}_\lambda(\cdot)^{-\eta}\|_\infty^{\frac{1}{q-1}} \text{ for some } c_{19} > 0 \\
&\leq \lambda c_{19}\|D\beta\|_\infty^{\frac{1}{q-1}} \text{ (see (19))}, \\
&\Rightarrow \bar{u}_\lambda \to 0 \text{ in } L^\infty(\Omega) \text{ as } \lambda \to 0^+.
\end{aligned}$$

We conclude that

$$\bar{u}_\lambda \to 0 \text{ in } W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega) \text{ as } \lambda \to 0^+.$$

The proof is now complete. □

We are ready to produce the first positive solution for problem (P). For that we do not need the asymptotic condition as $x \to +\infty$ on $f(z,\cdot)$(see hypothesis H_1(ii)).

Proposition 3.6. *If hypotheses H_0 and $H_1(i),(iii)$ hold, then problem (P) has a solution*

$$u_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0.$$

Proof. 3.5, for $\lambda \in (0,1)$ small, we have

$$\|\bar{u}_\lambda\|_\infty \le \delta, \tag{20}$$

with $\delta \in (0,1)$ as postulated by hypothesis H_1(iii). We have

$$-\Delta_p^a \bar{u}_\lambda - \Delta_q \bar{u}_\lambda \le \lambda\beta(z)\bar{u}_\lambda^{-\eta} + f(z,\bar{u}_\lambda) \text{ in } \Omega. \tag{21}$$

Fix such a $\lambda \in (0,1)$ and for notational simplicity let $\bar{u} = \bar{u}_\lambda$. From (20) and hypothesis H_1(iii), we see that $0 \le \bar{u}(z) < \gamma_0$ for a.a. $z \in \Omega$. So, we can introduce the Carathéodory function $\hat{m}(z,x)$ defined by

$$\hat{m}(z,x) = \begin{cases} \beta(z)\bar{u}(z)^{-\eta} + f(z,\bar{u}(z)) & \text{if } x < \bar{u}(z), \\ \beta(z)x^{-\eta} + f(z,x) & \text{if } \bar{u}(z) \le x \le \gamma_0, \\ \beta(z)\gamma_0^{-\eta} + f(z,\gamma_0) & \text{if } \gamma_0 < x. \end{cases} \tag{22}$$

We set $\hat{M}(z,x) = \int_0^x \hat{m}(z,s)\,ds$ and consider the functional $\hat{\psi} : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{\psi}(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{M}(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

From (18) and Proposition 3 of Papageorgiou-Smyrlis [30], we have that $\hat{\psi} \in C^1(W_0^{1,\vartheta}(\Omega))$. From (22) and Proposition 2.2, we see that $\hat{\psi}(\cdot)$ is coercive. Moreover, using Proposition 2.1 and the sequential weak lower semicontinuity of the modular function, we show that $\hat{\psi}(\cdot)$ is sequentially weakly lower semicontinuous. Then by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,\vartheta}(\Omega)$ such that

$$\hat{\psi}(u_0) = \inf\left\{\hat{\psi}(u) : u \in W_0^{1,\vartheta}(\Omega)\right\},$$

$$\Rightarrow \langle \hat{\psi}'(u_0), h\rangle = 0 \text{ for all } h \in W_0^{1,\vartheta}(\Omega). \tag{23}$$

In (23), first we choose the test function $h = (\bar{u} - u_0)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned} &\langle V(u_0), (\bar{u} - u_0)^+\rangle \\ &= \int_\Omega [\beta(z)\bar{u}^{-\eta} + f(z,\bar{u})](\bar{u} - u_0)^+\,dz \text{ (see (22))} \\ &\ge \langle V(\bar{u}), (\bar{u} - u_0)^+)\rangle \text{ (see (21))}, \\ &\Rightarrow \bar{u} \le u_0 \text{ (see Proposition 2.3)}. \end{aligned}$$

Next in (23), we choose the test function $h = (u_0 - \gamma_0)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned} &\langle V(u_0), (u_0 - \gamma_0)^+\rangle \\ &= \int_\Omega \left[\beta(z)\gamma_0^{-\eta} + f(z,\gamma_0)\right](u_0 - \gamma_0)^+\,dz \text{ (see (22))} \\ &\le 0 = \langle V(\gamma_0), (u_0 - \gamma_0)^+\rangle \text{ (see hypothesis } H_1\text{(iii))}, \\ &\Rightarrow u_0 \le \gamma_0. \end{aligned}$$

We have established that

$$\bar{u} \le u_0 \le \gamma_0 \text{ in } \Omega.$$

$$\Rightarrow u_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0 \text{ is a solution of } (P),$$

which completes the proof. □

To produce a second solution for (P), we introduce the Carathéodory function $m(z,x)$, defined by

$$m(z,x) = \begin{cases} \beta(z)\bar{u}(z)^{-\eta} + f(z,\bar{u}(z)) & \text{if } x < \bar{u}(z), \\ \beta(z)x^{-\eta} + f(z,x) & \text{if } \bar{u}(z) \le x. \end{cases} \tag{24}$$

We set $M(z,x) = \int_0^x m(z,s)\,ds$ and consider the C^1-functional $\psi : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega M(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

In what follows, $u_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ is the solution of (P) produced in Proposition 3.6.

Proposition 3.7. *If hypotheses H_0 and H_1 hold, then $C_k(\psi, u_0) = C_k(\hat{\psi}, u_0)$ for all $k \in \mathbb{N}_0$.*

Proof. Let $d \ge \gamma_0$ and define

$$m_d(z,x) = \begin{cases} m(z,x) & \text{if } x \le d, \\ m(z,d) & \text{if } d < x. \end{cases} \tag{25}$$

We set $M_d(z,x) = \int_0^x m_d(z,s)\,ds$ and consider the C^1-functional $\psi_d : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi_d(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega M_d(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

For $u \in W_0^{1,\vartheta}(\Omega)$, we have

$$\begin{aligned} |\psi_d(u) - \hat{\psi}(u)| &\le \int_\Omega |M_d(z,u) - \hat{M}(z,u)|\,dz \\ &\le \int_\Omega |M_d(z,u) - M_d(z,u_0)|\,dz + \int_\Omega |\hat{M}(z,u_0) - \hat{M}(z,u)|\,dz \\ &\quad \text{(since } 0 \le u_0 \le \gamma_0 < d, \text{see (22), (24), (25)).} \end{aligned} \tag{26}$$

Let $\Omega_- = \{z \in \Omega : u(z) < \bar{u}(z)\}$. In Ω, we have

$$M_d(z,u) = M(z,u) = \hat{M}(z,u),\ M_d(z,u_0) = M(z,u_0) = \hat{M}(z,u_0) \text{ (see (22), (24), (25)).}$$

Then if follows from in Ω_-, we can say that

$$|M_d(z,u) - M_d(z,u_0)| = |\hat{M}(z,u) - \hat{M}(z,u_0)| = |M(z,u) - M(z,u_0)|. \tag{27}$$

Using (24), (25), in Ω_-, we have

$$
\begin{aligned}
&|M(z,u)-M(z,u_0)| \\
&\leq |\beta(z)\bar{u}^{-\eta}u-\beta(z)\bar{u}^{1-\eta}|+\frac{\beta(z)}{1-\eta}(u_0^{1-\eta}-\bar{u}^{1-\eta}) \\
&\quad +f(z,\bar{u})|u-\bar{u}|+|F(z,u_0)-F(z,\bar{u})| \\
&=\beta(z)\bar{u}^{-\eta}(\bar{u}-u)+\frac{\beta(z)}{1-\eta}(u_0^{1-\eta}-\bar{u}^{1-\eta})+c_{20}(u_0-\bar{u}) \text{ for some } c_{20}>0 \\
&\leq \beta(z)\bar{u}^{-\eta}(u_0-u)+\frac{\beta(z)}{1-\eta}(u_0^{1-\eta}-\bar{u}u_0^{-\eta})+c_{20}(u_0-\bar{u}) \\
&\quad (\text{since in } \Omega_-, \bar{u}^{1-\eta}=\bar{u}\cdot\bar{u}^{-\eta}\geq \bar{u}\cdot u_0^{-\eta}) \\
&=\beta(z)\bar{u}^{-\eta}(u_0-u)+\frac{\beta(z)}{1-\eta}u_0^{-\eta}(u_0-\bar{u})+c_{20}(u_0-\bar{u}) \\
&\leq c_{21}(u_0-u) \text{ in } \Omega_- \text{ for some } c_{21}>0 \text{ (see Proposition 3.4)}.
\end{aligned}
\tag{28}
$$

Next let $\Omega_0=\{z\in\Omega : \bar{u}(z)\leq u(z)\leq \gamma_0\}$. We see that in Ω_0

$$
M(z,u)=\hat{M}(z,u),\ M(z,u_0)=\hat{M}(z,u_0) \text{ (see (22), (24), (25))}. \tag{29}
$$

In Ω_0, we have

$$
\begin{aligned}
&|M(z,u)-M(z,u_0)| \\
&=\frac{\beta(z)}{1-\eta}|u^{1-\eta}-u_0^{1-\eta}|+|F(z,u)-F(z,u_0)| \\
&\leq c_{22}|u-u_0| \text{ in } \Omega_0, \text{for some } c_{22}>0.
\end{aligned}
\tag{30}
$$

Here we have used the fact that $x\mapsto x^{1-\eta}(x\geq 0)$is a concave function, thus locally Lipschitz. Also, note that $F(z,\cdot)$ is $L^\infty(\Omega)$-locally Lipschitz.

Let $\Omega_+^1=\{z\in\Omega : \gamma_0<u(z)\leq d\}$. Then in Ω_+^1, we have

$$
\begin{aligned}
&|M(z,u)-M(z,u_0)| \\
&\leq \left|\frac{\beta(z)}{1-\eta}(u^{1-\eta}-\bar{u}_\lambda^{1-\eta})-\frac{\beta(z)}{1-\eta}(u_0^{1-\eta}-\bar{u}_\lambda^{1-\eta})\right|+|F(z,u)-F(z,u_0)| \\
&\leq \frac{\beta(z)}{1-\eta}(u^{1-\eta}-u_0^{1-\eta})+|F(z,u)-F(z,u_0)| \text{ (note that } u_0<u \text{ in } \Omega_+^1) \\
&\leq c_{23}(u-u_0) \text{ in } \Omega_+^1,\ \text{for some } c_{23}>0.
\end{aligned}
\tag{31}
$$

Moreover, again in Ω_+^1, we have

$$
\begin{aligned}
&|\hat{M}(z,u)-\hat{M}(z,u_0)| \\
&\leq \left|\frac{\beta(z)}{1-\eta}\gamma_0^{1-\eta}+\beta(z)\gamma_0^{-\eta}(u-\gamma_0)-\frac{\beta(z)}{1-\eta}u_0^{1-\eta}\right|+|F(z,u)-F(z,u_0)| \\
&\leq c_{24}(u-u_0) \text{ in } \Omega_+^1,\ \text{for some } c_{24}>0.
\end{aligned}
\tag{32}
$$

Finally let $\Omega_+^2 = \{z \in \Omega : d < u(z)\}$. In Ω_+^2, we have

$$\begin{aligned} &|M(z,u) - M(z,u_0)| \\ &\leq \left| \frac{\beta(z)}{1-\eta} d^{1-\eta} + \beta(z) d^{-\eta}(u-d) - \frac{\beta(z)}{1-\eta} u_0^{1-\eta} \right| \\ &\quad + |F(z,d) + f(z,d)(u-d) - F(z,u_0)| \\ &\leq c_{25}(u-u_0) \text{ in } \Omega_+^2, \text{ for some } c_{25} > 0. \end{aligned} \tag{33}$$

Still in Ω_+^2, we have

$$\begin{aligned} &|\hat{M}(z,u) - \hat{M}(z,u_0)| \\ &\leq \left| \frac{\beta(z)}{1-\eta} \gamma_0^{1-\eta} + \beta(z) \gamma_0^{-\eta}(u-\gamma_0) - \frac{\beta(z)}{1-\eta} u_0^{1-\eta} \right| \\ &\quad + |F(z,\gamma_0) + f(z,\gamma_0)(u-\gamma_0) - F(z,u_0)| \\ &\leq c_{26}(u-u_0) \text{ in } \Omega_+^2, \text{ for some } c_{26} > 0. \end{aligned} \tag{34}$$

We return to (26) and use (27)-(34). We obtain

$$|\psi_d(u) - \hat{\psi}(u)| \leq c_{27}\|u-u_0\| \text{ for some } c_{27} > 0. \tag{35}$$

Also for every $u, h \in W_0^{1,\vartheta}(\Omega)$, we have

$$\begin{aligned} &|\langle \psi_d'(u) - \hat{\psi}'(u), h \rangle| \\ &\leq \int_\Omega |m_d(z,u) - \hat{m}(z,u)||h|\,dz \\ &\leq \int_\Omega |m_d(z,u) - m_d(z,u_0)||h|\,dz + \int_\Omega |\hat{m}(z,u_0) - \hat{m}(z,u)||h|\,dz \\ &\quad (\text{since } u_0 \leq \gamma_0 < d, \text{see } (22), (24), (25)) \\ &\leq c_{28}\|u-u_0\|\|h\| \text{ for some } c_{28} > 0 \\ &\quad (\text{note that } m_d(z,\cdot), \hat{m}(z,\cdot) \text{ are } L^\infty(\Omega)\text{-locally Lipschitz}), \end{aligned}$$

$$\Rightarrow \|\psi_d'(u) - \hat{\psi}'(u)\| \leq c_{28}\|u-u_0\|. \tag{36}$$

Then from (35) and (36), we see that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$\|\psi_d - \hat{\psi}\|_{C^1(\bar{B}_\delta(u_0))} \leq \epsilon,$$

where $\bar{B}_\delta(u_0) = \left\{ u \in W_0^{1,\vartheta}(\Omega) : \|u-u_0\| \leq \delta \right\}$. Then the C^1-continuity property of critical groups (see Hu-Papageorgiou [17], p.179), implies that

$$C_k(\psi_d, u_0) = C_k(\hat{\psi}, u_0) \text{ for all } k \in \mathbb{N}_0.$$

Let $d \to \infty$ and use Theorem D.6, p.615 of Granas-Dugundji [10], to conclude that $C_k(\psi, u_0) = C_k(\hat{\psi}, u_0)$ for all $k \in \mathbb{N}_0$. □

Remark 3.8. We outline an alternative way to prove this proposition. Let $V = W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$. Evidently $V \hookrightarrow W_0^{1,\vartheta}(\Omega)$. The estimates in the previous proof remain valid because the functions in V are bounded. Therefore

$$C_k(\psi|_V, u_0) = C_k(\hat{\psi}|_V, u_0) \quad \text{for all } k \in \mathbb{N}_0 \ (C^1\text{-continuity property of critical groups}).$$

Using Galerkin approximations and the extended Gromoll-Meyer theory of Chang-Ghoussoub [6], we show that

$$C_k(\psi|_V, u_0) = C_k(\psi, u_0) \text{ and } C_k(\hat{\psi}|_V, u_0) = C_k(\hat{\psi}, u_0) \text{ for all } k \in \mathbb{N}_0.$$

Therefore we conclude that $C_k(\psi, u_0) = C_k(\hat{\psi}, u_0)$ for all $k \in \mathbb{N}_0$(see also Liu-Wu [23]).

Proposition 3.9. *If hypotheses H_0 and H_1 hold, then the functional $\psi(\cdot)$ satisfies the C-condition.*

Proof. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ such that

$$|\psi(u_n)| \le c_{29} \text{ for some } c_{29} > 0, \text{ all } n \in \mathbb{N}, \tag{37}$$

$$(1 + \|u_n\|)\psi'(u_n) \to 0 \text{ in } W_0^{1,\vartheta}(\Omega)^* \text{ as } n \to \infty. \tag{38}$$

From (38), we have

$$\left| \langle V(u_n), h\rangle - \int_\Omega m(z, u_n) h\, dz \right| \le \frac{\epsilon_n \|h\|}{1 + \|u_n\|}, \tag{39}$$

for all $h \in W_0^{1,\vartheta}(\Omega)$, all $n \in \mathbb{N}$, with $\epsilon_n \to 0^+$. In (39), we choose the test function $h = -u_n^- \in W_0^{1,\vartheta}(\Omega)$. Then

$$\rho_\vartheta(Du_n^-) \le \epsilon_n \text{ for all } n \in \mathbb{N},$$

$$\Rightarrow u_n^- \to 0 \text{ in } W_0^{1,\vartheta}(\Omega), \text{ as } n \to \infty. \tag{40}$$

Claim: $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded.

We argue by contradiction. So, suppose that at least for a subsequence, we have

$$\|u_n^+\| \to +\infty. \tag{41}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}, n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\vartheta}(\Omega), y_n \to y \text{ in } L^r(\Omega), y \ge 0 \text{ (see Proposition 2.1)}. \tag{42}$$

First assume that $y \ne 0$ and define $\hat{\Omega} = \{z \in \Omega : y(z) > 0\}$. Then $|\hat{\Omega}|_N > 0$(see (42)). On account of (41), we see that

$$u_n^+(z) \to +\infty \text{ for a.a. } z \in \hat{\Omega}. \tag{43}$$

Then we have

$$\frac{F(z, u_n^+(z))}{\|u_n^+\|^p} = \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \to +\infty \text{ for a.a. } z \in \hat{\Omega} \text{ (see (43) and hypothesis } \mathrm{H}_1\text{(ii))}.$$

Using Fatou's lemma, we obtain

$$\lim_{n\to\infty} \int_{\hat{\Omega}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz = +\infty. \tag{44}$$

Note that hypotheses H_1(i), (ii) imply that

$$F(z, x) \ge -c_{30} \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0, \text{ some } c_{30} > 0. \tag{45}$$

Therefore, we have

$$\begin{aligned}\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz &= \int_{\hat\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz + \int_{\Omega\setminus\hat\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz \\ &\geq \int_{\hat\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz - \frac{c_{30}}{\|u_n^+\|^p}|\Omega|_N \text{ (see 45)},\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty}\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz = +\infty \text{ (see (44))}. \tag{46}$$

On the other hand from (37), (40), (41)(recall $q<p$), (24) and Proposition 3.4, it follows that

$$\left|\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz\right| \leq c_{31} + \frac{1}{p}\rho_a(Dy_n) \text{ for some } c_{31}>0, \text{ all } n\in\mathbb{N},$$

$$\begin{aligned}&\Rightarrow \left|\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz\right| \leq c_{32} \text{ for some } c_{32}>0, \text{ all } n\in\mathbb{N} \\ &\quad\text{(we use that } \|y_n\|=1 \text{ for all } n\in\mathbb{N} \text{ and Proposition 2.2).}\end{aligned} \tag{47}$$

Comparing (46) and (47), we have a contradiction.

Now assume that $y=0$(see (42)). From (39), (40) and since $q<p$, we have

$$\left|\int_\Omega \frac{m(z,u_n^+)}{\|u_n^+\|^{p-1}}h\,dz\right| \leq \epsilon_n'\|h\| + \left|\int_\Omega a(z)|Dy_n|^{p-2}(Dy_n,Dh)_{\mathbb{R}^N}\,dz\right|$$

for all $h\in W_0^{1,\vartheta}(\Omega)$, all $n\in\mathbb{N}$, with $\epsilon_n'\to 0^+$.

Using Hölder's inequality and the fact that $\|y_n\|=1$ for all $n\in\mathbb{N}$, we obtain

$$\left|\int_\Omega a(z)|Dy_n|^{p-2}(Dy_n,Dh)_{\mathbb{R}^N}\,dz\right| \leq c_{33}\|h\|$$

for some $c_{33}>0$, all $h\in W_0^{1,\vartheta}(\Omega)$, all $n\in\mathbb{N}$. Therefore we can write that

$$\left|\int_\Omega \frac{m(z,u_n^+)}{\|u_n^+\|^{p-1}}h\,dz\right| \leq c_{34}\|h\| \tag{48}$$

for some $c_{34}>0$, all $h\in W_0^{1,\vartheta}(\Omega)$, all $n\in\mathbb{N}$.

Consider the linear functional $i_n: W_0^{1,\vartheta}(\Omega)\mapsto\mathbb{R}$, $n\in\mathbb{N}$, defined by

$$i_n(h) = \int_\Omega \frac{m(z,u_n^+)}{\|u_n^+\|^{p-1}}h\,dz \text{ for all } h\in W_0^{1,\vartheta}(\Omega).$$

From (48), we see that

$$i_n\in W_0^{1,\vartheta}(\Omega)^*,\ \|i_n\|_*\leq c_{34} \text{ for all } n\in\mathbb{N}. \tag{49}$$

Since $W_0^{1,\vartheta}(\Omega)\hookrightarrow L^r(\Omega)$(since $r<q^*$, see Proposition 2.1), using the Hahn-Banach theorem (see [32], p.210), we can find $\hat\imath_n\in L^r(\Omega)^*$ such that

$$\hat\imath_n|_{W_0^{1,\vartheta}(\Omega)} = i_n,\ \|\hat\imath_n\|_{L^r(\Omega)^*}\leq c_{34} \text{ for all } n\in\mathbb{N} \text{ (see (49))}. \tag{50}$$

The Riesz representation theorem for the Lebesgue spaces, says that there exists unique $\hat{g}_n \in L^{r'}(\Omega)(r' = \frac{r}{r-1})$ such that

$$\begin{cases} \hat{\imath}_n(h) = \int_\Omega \hat{g}_n h \, dz \text{ for all } h \in L^r(\Omega), \text{ all } n \in \mathbb{N}, \\ \|\hat{g}_n\|_{r'} \leq c_{34} \text{ for all } n \in \mathbb{N} (\text{see } (50)). \end{cases} \tag{51}$$

From (50), (51), choosing $h = y_n \in W_0^{1,\vartheta}(\Omega) \hookrightarrow L^r(\Omega)$ and recalling that $y = 0$, we obtain

$$\int_\Omega \frac{m(z, u_n^+)}{\|u_n^+\|^{p-1}} y_n \, dz \to 0. \tag{52}$$

We return to (39), multiply the inequality with $\frac{1}{\|u_n^+\|^{p-1}}$ and choose the test function $h = y_n \in W_0^{1,\vartheta}(\Omega)$. Since $q < p$ and using (52), we obtain

$$\rho_a(Dy_n) \to 0 \text{ as } n \to +\infty. \tag{53}$$

Recall that $\|y_n\| = 1$. Therefore

$$\rho_\vartheta(Dy_n) = \rho_a(Dy_n) + \|Dy_n\|_q^q = 1 \text{ for all } n \in \mathbb{N} \text{ (see Proposition 2.2)},$$

$$\Rightarrow \|Dy_n\|_q \to 1 \text{ as } n \to \infty \text{ (see (53))},$$

$$\Rightarrow \|Du_n^+\|_q \to +\infty \text{ (see (41))}. \tag{54}$$

In (39), we choose the test function $h = u_n^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$-\rho_a(Du_n^+) - \|Du_n^+\|_q^q + \int_\Omega m(z, u_n^+) u_n^+ \, dz \leq \epsilon_n \text{ for all } n \in \mathbb{N}. \tag{55}$$

On the other hand, from (38) and (40), we have

$$\rho_a(Du_n^+) + \frac{p}{q}\|Du_n^+\|_q^q - \int_\Omega pM(z, u_n^+) \, dz \leq c_{35} \tag{56}$$

for some $c_{35} > 0$, all $n \in \mathbb{N}$. We add (55) and (56) and obtain

$$\left(\frac{p}{q} - 1\right) \|Du_n^+\|_q^q + \int_\Omega \left[m(z, u_n^+)u_n^+ - pM(z, u_n^+)\right] dz \leq c_{36}$$

for some $c_{36} > 0$, all $n \in \mathbb{N}$,

$$\Rightarrow \left(\frac{p}{q} - 1\right) \|Du_n^+\|_q^q + \int_\Omega \left[f(z, u_n^+)u_n^+ - pF(z, u_n^+)\right] dz$$

$$\leq c_{37} \left[1 + \|Du_n^+\|_q^\tau + \|_q^{1-\eta}\right] \text{ for some } c_{37} > 0, \text{ all } n \in \mathbb{N}$$

(see (24) and Theorem 13.17, p.196, in Hewitt-Stromberg [16]),

$$\Rightarrow \|Du_n^+\|_q^q \leq c_{38} \left[1 + \|Du_n^+\|_q^\tau\right]$$

for some $c_{38} > 0$, all $n \in \mathbb{N}$(see hypothesis H_1(ii) and recall $0 < \eta < 1 < q$),

$$\Rightarrow \|Du_n^+\|_q \leq c_{39} \text{ for some } c_{39} > 0, \text{ all } n \in \mathbb{N} (\text{since } \tau < q). \tag{57}$$

We compare (54) and (57) and have a contradiction. Therefore $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded. This proves the "**Claim**".

The **Claim** and (40), imply that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega) \text{ is bounded.}$$

So, we assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\vartheta}(\Omega),\ u_n \to u \text{ in } L^r(\Omega). \tag{58}$$

In (39), we choose the test function $h = u_n - u \in W_0^{1,\vartheta}(\Omega)$, pass to the limit as $n \to \infty$ and use (58). Then

$$\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0,$$
$$\Rightarrow u_n \to u \text{ in } W_0^{1,\vartheta}(\Omega) \text{ (see Proposition 2.3)}.$$

Therefore $\psi(\cdot)$ satisfies the C-condition. □

Proposition 3.10. *If hypotheses H_0 and H_1 hold, then u_0 is a local minimizer of the functional $\psi(\cdot)$.*

Proof. From the proof of Proposition 3.6, we know that u_0 is a global minimizer of $\hat{\psi}(\cdot)$. Therefore

$$C_k(\hat{\psi}, u_0) = \delta_{k,0}\mathbb{R} \text{ (see [28],p.477)},$$
$$\Rightarrow C_k(\psi, u_0) = \delta_{k,0}\mathbb{R} \text{ (see Proposition 3.7)}.$$

Then Theorem 4.6, p.43, of Chang [5], implies that u_0 is a local minimizer of $\psi(\cdot)$. □

In what follows,

$$[\bar{u}) = \left\{ h \in W_0^{1,\vartheta} : \bar{u}(z) \le h(z) \text{ for a.a. } z \in \Omega \right\}.$$

Proposition 3.11. *If hypotheses H_0 and H_1 hold, then $K_\psi \subseteq [\bar{u}) \cap L^\infty(\Omega)$.*

Proof. Let $u \in K_\psi$. We have

$$\langle V(u), h\rangle = \int_\Omega m(z,u)h\,dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega). \tag{59}$$

In (59), we choose the test function $h = (\bar{u} - u)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned}\langle V(u), (\bar{u}-u)^+\rangle &= \int_\Omega \left[\beta(z)\bar{u}^{-\eta} + f(z,\bar{u})\right](\bar{u}-u)^+\,dz \text{ (see (24))}\\ &\ge \int_\Omega \beta(z)\bar{u}^{-\eta}(\bar{u}-u)^+\,dz \text{ (see hypothesis H}_1\text{(iii))}\\ &= \langle V(\bar{u}), (\bar{u}-u)^+)\rangle \text{ (see Proposition 3.3)},\\ &\Rightarrow \bar{u} \le u \text{(see Proposition 2.3)}.\end{aligned}$$

Moreover, from Proposition 2.6 we have $u \in L^\infty(\Omega)$. Therefore

$$K_\psi \subseteq [\bar{u}) \cap L^\infty(\Omega).$$

The proof is now complete. □

Now we can produce a second position solution for problem (P) and have the first multiplicity theorem.

Theorem 3.12. *If hypotheses H_0 and H_1 hold, then problem (P) has at least two solutions*

$$u_0, \hat{u} \in W_0^{1,\vartheta}(\Omega),\ 0 \prec u_0,\ 0 \prec \hat{u}.$$

Proof. From Proposition 3.4, we already have one solution

$$u_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0.$$

On account of Proposition 3.11, we may assume that

$$K_\psi \text{ is finite.} \tag{60}$$

Otherwise, we already have an infinity of bounded solutions of (P) and so we are done. Then from (60), Proposition 3.9, 3.10 and Theorem 5.7.6, p.449 of Papageorgiou-Rădulescu-Repovš [28], we can find $\rho \in (0,1)$ small such that

$$\psi(u_0) < \inf\{\psi(u) : \|u - u_0\| = \rho\} = m_+. \tag{61}$$

If $u \in C_0^1(\Omega)$ with $u(z) > 0$ for all $z \in \Omega$, then on account of hypothesis H_1(ii), we have

$$\psi(tu) \to -\infty \text{ as } t \to +\infty. \tag{62}$$

Then Proposition 3.9, (61), (62) and the mountain pass theorem, imply that we can find $\hat{u} \in W_0^{1,\vartheta}(\Omega)$ such that

$$\begin{cases} \hat{u} \in K_\psi \subseteq [\bar{u}) \cap L^\infty(\Omega), (\text{see Proposition } 3.11), \\ m_+ \le \psi(\hat{u}). \end{cases} \tag{63}$$

Therefore $\hat{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec \hat{u}$ is a second bounded solution of (P) and $\hat{u} \neq u_0$(see (61), (63)). □

4. Problem $(P_\lambda)^1$

In this section, we focus on problem $(P_\lambda)^1$. For this problem, we prove an existence and multiplicity result which is global in the parameter $\lambda > 0$(a bifurcation-type result).

We impose the following conditions on the perturbation $f(z,x)$.

(H_2) : $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is an $L^\infty(\Omega)$-locally Lipschitz function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \le \hat{a}(z)[1 + x^{r-1}]$ for a.a. $z \in \Omega$, all $x \ge 0$, with $\hat{a} \in L^\infty(\Omega)$ and $p < r < q^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exist $\hat{c} > 0$ and $\tau \in [1,q)$ such that

$$-\hat{c}(1 + x^\tau) \le f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega,\ \text{all } x \ge 0;$$

(iii) there exist $\delta > 0$ and $l \in L^\infty(\Omega)_+$ such that

$$0 \le f(z,x) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \le x \le \delta,$$

$$-\hat{c}_\delta(1 + x^{d-1}) \le f(z,x) \text{ for a.a. } z \in \Omega,\ \text{all } x \ge \delta,\ \text{with } \hat{c}_\delta > 0, d > 1,$$

$$l(z) \le \hat{\lambda}_1(q) \text{ for a.a. } z \in \Omega,\ l \not\equiv \hat{\lambda}_1(q) \text{ and}$$

$$\limsup_{x\to 0^+} \frac{qF(z,x)}{x^q} \le l(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 4.1. Again we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Note that hypotheses H_2(i), (ii) are the same as the corresponding hypotheses H_1(i), (ii). The conditions near zero differ (see hypotheses H_2(iii) and H_1(iii)). Also, we have added a mild local perturbed monotonicity condition (see hypothesis H_2(iv)). We point out that we do not assume that $f \geq 0$ (as is the case in Liu-Papageorgiou [21], where $f(z,x) = \zeta(z)x^{p-1}$ for a.a. $z \in \Omega$, all $x \geq 0$ with $\zeta \in L^\infty(\Omega)_+$) or that $f(z,\cdot)$ changes sign (as is the case in Han-Liu-Papageorgiou [14], see hypothesis H_1(iii)). So, hypotheses H_2, unify the works of [14] and [21].

We introduce the following two sets:

$$\mathcal{L}_1 = \left\{\lambda > 0 : \text{problem } (P_\lambda)^1 \text{ has a solution}\right\},$$

$$S_\lambda^1 = \text{solution set of } (P_\lambda)^1.$$

Let $\bar{u}_\lambda \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega), 0 \prec \bar{u}_\lambda$ be the unique solution of problem (9) (see Proposition 3.3). We know that

$$\bar{u}_\lambda \to 0 \text{ in } W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega) \text{ as } \lambda \to 0^+.$$

Therefore, we can find $\lambda_0 > 0$ such that

$$\|\bar{u}_\lambda\|_\infty \leq \delta \text{ for all } 0 < \lambda \leq \lambda_0, \tag{64}$$

with $\delta > 0$ as postulated by hypothesis H_2(iii). Then we introduce the Carathéodory function $g_\lambda(z,x)(0 < \lambda \leq \lambda_0)$ defined by

$$g_\lambda(z,x) = \begin{cases} \lambda\beta(z)\bar{u}_\lambda(z)^{-\eta} + f(z,\bar{u}_\lambda(z)) & \text{if } x < \bar{u}_\lambda(z), \\ \lambda\beta(z)x^{-\eta} + f(z,x) & \text{if } \bar{u}_\lambda(z) \leq x. \end{cases} \tag{65}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)\,ds$ and consider the C^1-functional $\varphi_\lambda : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega G_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

Proposition 4.2. *If hypotheses H_0 and H_2 hold, then $\mathcal{L}_1 \neq \emptyset$ and for every $\lambda \in \mathcal{L}_1$, we have*

$$\emptyset \neq S_\lambda^1 \subseteq W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u \textit{ for all } u \in S_\lambda^1.$$

Proof. Hypotheses H_2(i), (iii) imply that given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$F(z,x) \leq \frac{1}{q}\left[l(z) + \epsilon\right]x^q + c_\epsilon x^r \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{66}$$

Let $u \in W_0^{1,\vartheta}(\Omega)$ with $\|u\| \leq 1$. Then we have $(0 < \lambda \leq \lambda_0)$

$$\begin{aligned}
\varphi_\lambda(u) \geq\ & \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q \\
& - \int_{\{u<\bar{u}_\lambda\}} \lambda\beta(z)\bar{u}_\lambda^{-\eta}\,dz - \int_{\{\bar{u}_\lambda\leq u\}} \lambda\beta(z)\bar{u}_\lambda^{1-\eta}\,dz - \int_{\{\bar{u}_\lambda\leq u\}} \frac{\lambda\beta(z)}{1-\eta}(u^{1-\eta} - \bar{u}_\lambda^{1-\eta})\,dz \\
& - \int_{\{u<\bar{u}_\lambda\}} f(z,\bar{u}_\lambda)u\,dz - \int_{\{\bar{u}_\lambda\leq u\}} f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz - \int_{\{\bar{u}_\lambda\leq u\}} (F(z,u) - F(z,\bar{u}_\lambda))\,dz \\
\geq\ & \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \frac{\lambda\beta(z)}{1-\eta}|u|^{1-\eta}\,dz - \int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz - \int_\Omega F(z,u)\,dz \\
\geq\ & \frac{1}{p}\rho_a(Du) + \frac{1}{q}\left[\|Du\|_q^q - \int_\Omega l(z)|u|^q\,dz - \frac{\epsilon}{\hat{\lambda}_q(q)}\|Du\|_q^q\right] \\
& - c_{40}\|u\|^r - \int_\Omega \frac{\lambda\beta(z)}{1-\eta}|u|^{1-\eta}\,dz - \int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz \text{ for some } c_{40} > 0 \text{ (see (66))}, \\
\geq\ & \frac{1}{p}\rho_a(Du) + \frac{1}{q}\left[c_{41} - \frac{\epsilon}{\hat{\lambda}_1(q)}\right]\|Du\|_q^q - c_{40}\|u\|^r \\
& - \int_\Omega \frac{\lambda\beta(z)}{1-\eta}|u|^{1-\eta}\,dz - \int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz \text{ for some } c_{41} > 0 \\
& \text{(see Hu-Papageorgiou [17], p.218).}
\end{aligned}$$

Choosing $\epsilon \in (0, \hat{\lambda}_1(q)\cdot c_{41})$, we obtain

$$\begin{aligned}
\varphi_\lambda(u) &\geq c_{42}\rho_\vartheta(Du) - c_{40}\|u\|^r - \lambda c_{43}\|u\|^{1-\eta} - \int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz \\
&\qquad \text{for some } c_{42} > 0, c_{43} > 0 \\
&\geq c_{42}\|u\|^p - c_{40}\|u\|^r - \lambda c_{43}\|u\|^{1-\eta} - \int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz \\
&\qquad \text{(since } \|u\| \leq 1, \text{see Proposition 2.2).}
\end{aligned} \tag{67}$$

Let $\rho = \|u\| \leq 1$ small so that

$$c_{42}\rho^p - c_{43}\rho^r = m_\rho > 0 \text{ (recall } p < r).$$

Since $\int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz \to 0^+$ as $\lambda_0 \to 0^+$, choose $\hat{\lambda} \in (0, \lambda_0]$ small that

$$m_\rho - \lambda c_{43}\rho^{1-\eta} - \int_\Omega f(z,\bar{u}_\lambda)\bar{u}_\lambda\,dz \geq \hat{m}_\rho > 0 \text{ for all } 0 < \lambda \leq \hat{\lambda}.$$

Therefore, we have

$$\varphi_\lambda(u) \geq \hat{m}_\rho > 0 = \varphi_\lambda(0) \text{ for all } \|u\| = \rho, \text{ all } 0 < \lambda \leq \hat{\lambda}. \tag{68}$$

From Proposition 3.9, we know that

$$\varphi_\lambda(\cdot) \text{ satisfies the C-condition.} \tag{69}$$

Moreover, if $u \in C_+$, $u(z) > 0$ for all $z \in \Omega$, then on account of hypothesis H_2(ii), we have

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{70}$$

Then (68), (69), (70) and the mountain pass theorem, imply that there exists $u_\lambda \in W_0^{1,\vartheta}(\Omega)$ such that

$$u_\lambda \in K_{\varphi_\lambda} \subseteq [\bar{u}_\lambda) \cap L^\infty(\Omega) \text{ (see Proposition 3.11)},$$

$$\varphi_\lambda(0) = 0 < \hat{m}_\rho \le \varphi_\lambda(u_\lambda)(0 < \lambda \le \hat{\lambda}).$$

It follows that

$$u_\lambda \in S^1_\lambda \text{ (see (65))},$$
$$\Rightarrow (0, \hat{\lambda}] \in \mathcal{L}_1 \neq \emptyset.$$

Moreover, from Proposition 2.6, we infer that

$$\emptyset \neq S^1_\lambda \subseteq W^{1,\vartheta}_0(\Omega) \cap L^\infty(\Omega) \text{ for all } \lambda \in \mathcal{L}_1.$$

If $u \in S^1_\lambda$, let $\rho = \|u\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv), then

$$-\Delta^a_p u - \Delta_q u + \hat{\xi}_\rho u^{p-1} \ge 0 \text{ in } \Omega.$$

Invoking Proposition 2.4 of Papageorgiou-Vetro-Vetro [31], we conclude that

$$0 \prec u \text{ for all } u \in S^1_\lambda.$$

This completes the proof. □

Next, we show that $\mathcal{L}_1$ is connected (an interval).

Proposition 4.3. *If hypotheses H_0 and H_2 hold, $\lambda \in \mathcal{L}_1$, and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}_1$.*

Proof. Let $u_\lambda \in S^1_\lambda$. Then $u_\lambda \in L^\infty(\Omega)$(see Proposition 2.6). For $\rho = \|u_\lambda\|_\infty$, let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv). We consider the following purely singular Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta^a_p u(z) - \Delta_q u(z) + \hat{\xi}_\rho u(z)^{p-1} = \lambda\beta(z)u(z)^{-\eta}, \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0,\ \lambda > 0. \end{array} \right\} \tag{71}$$

We know (see Proposition 3.3 and also Proposition 3.1 of Bai-Papageorgiou-Zeng [3]), that problem (71) has a unique solution $\tilde{u} \in W^{1,\vartheta}_0(\Omega) \cap L^\infty(\Omega), 0 \prec \tilde{u}_\lambda$, $\{\tilde{u}_\lambda\}_{\lambda>0}$ is nondecreasing. Note that

$$\begin{aligned} &-\Delta^a_p u_\lambda - \Delta_q u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} \\ &= \lambda\beta(z)u_\lambda^{-\eta} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \text{ (since } u_\lambda \in S^1_\lambda) \\ &\ge \lambda\beta(z)u_\lambda^{-\eta} \text{in } \Omega \text{ (see hypothesis } H_2\text{(iv))}. \end{aligned} \tag{72}$$

We introduce the Carathéodory function $\hat{e}_\lambda(z, x)$ defined by

$$\hat{e}_\lambda(z, x) = \begin{cases} \lambda\beta(z)x^{-\eta} & \text{if } 0 < x \le u_\lambda(z) \\ \lambda\beta(z)u_\lambda(z)^{-\eta} & \text{if } u_\lambda(z) < x. \end{cases} \tag{73}$$

Then we consider the purely singular Dirichlet problem

$$-\Delta^a_p u(z) - \Delta_q u(z) + \hat{\xi}_\rho u(z)^{p-1} = \hat{e}_\lambda(z, u(z)) \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ u > 0. \tag{74}$$

As in the proof of Proposition 3.1 of [3], using regularizations and a fixed point argument, we show that problem (74) has a solution $\tilde{u}^*_\lambda \in W^{1,\vartheta}_0(\Omega) \cap L^\infty(\Omega)$. Hence

$$\langle V(\tilde{u}^*_\lambda), h\rangle + \int_\Omega \hat{\xi}_\rho(\tilde{u}^*_\lambda)^{p-1}h\,dz = \int_\Omega \hat{e}_\lambda(z, \tilde{u}^*_\lambda)h\,dz \text{ for all } h \in W^{1,\vartheta}_0(\Omega).$$

We choose the test function $h = (\tilde{u}_\lambda^* - u_\lambda)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned}
&\langle V(\tilde{u}_\lambda^*), (\tilde{u}_\lambda^* - u_\lambda)^+\rangle + \int_\Omega \hat{\xi}_\rho(\hat{u}_\lambda^* - u_\lambda)^+ \, dz \\
&= \int_\Omega \lambda\beta(z) u_\lambda^{-\eta}(\tilde{u}_\lambda^* - u_\lambda)^+ \, dz (\text{see } (73)) \\
&\leq \langle V(u_\lambda), (\tilde{u}_\lambda^* - u_\lambda)^+\rangle + \int_\Omega \hat{\xi}_\rho u_\lambda^{p-1}(\tilde{u}_\lambda^* - u_\lambda)^+ \, dz \text{ (see (72))}, \\
&\qquad \Rightarrow \tilde{u}_\lambda^* \leq u_\lambda \text{ (see Proposition 2.3)}.
\end{aligned} \tag{75}$$

From (73), (75), we infer that $\tilde{u}_\lambda^*$ is a solution of (71). Then the uniqueness of the solution of (71), implies that

$$\tilde{u}_\lambda^* = \tilde{u}_\lambda \Rightarrow \tilde{u}_\mu \leq \tilde{u}_\lambda \leq u_\lambda (\text{recall that } \{\tilde{u}_\lambda\}_{\lambda>0} \text{ is nondecreasing}). \tag{76}$$

Using (75), we can introduce the Carathéodory function $\hat{k}_\mu(z,x)$ defined by

$$\hat{k}_\mu(z,x) = \begin{cases} \mu\beta(z)\tilde{u}_\mu(z)^{-\eta} + f(z,\tilde{u}_\mu(z)) + \hat{\xi}_\rho \tilde{u}_\mu(z)^{p-1} & \text{if } x < \tilde{u}_\mu(z), \\ \mu\beta(z)x^{-\eta} + f(z,x) + \hat{\xi}_\rho x^{p-1} & \text{if } \tilde{u}_\mu(z) \leq x \leq u_\lambda(z), \\ \mu\beta(z)u_\lambda(z)^{-\eta} + f(z,u_\lambda(z)) + \hat{\xi}_\rho u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{77}$$

We set $\hat{K}_\mu(z,x) = \int_0^x \hat{k}_\mu(z,s)\, ds$ and consider the C^1-functional $\hat{\sigma}_\mu : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{\sigma}_\mu(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q + \frac{\hat{\xi}_\rho}{p}\|u\|_p^p - \int_\Omega \hat{K}_\mu(z,u)\, dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

From (77) and Proposition 2.2, it is clear that $\hat{\sigma}_\mu(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous (see Proposition 2.1). So, we can find $u_\mu \in W_0^{1,\vartheta}(\Omega)$ such that

$$\begin{aligned}
&\hat{\sigma}_\mu(u_\mu) = \inf\left\{\hat{\sigma}_\mu(u) : u \in W_0^{1,\vartheta}(\Omega)\right\}, \\
&\Rightarrow \langle \hat{\sigma}_\mu'(u_\mu), h\rangle = 0 \text{ for all } h \in W_0^{1,\vartheta}(\Omega), \\
&\Rightarrow \langle V(u_\mu), h\rangle + \int_\Omega \hat{\xi}_\rho |u_\mu|^{p-2}u_\mu h\, dz = \int_\Omega \hat{k}_\mu(z,u_\mu)h\, dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega).
\end{aligned} \tag{78}$$

In (78), we choose the test function $h = (\tilde{u}_\mu - u_\mu)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned}
&\langle V(u_\mu), (\tilde{u}_\mu - u_\mu)^+\rangle + \int_\Omega \hat{\xi}_\rho |u_\mu|^{p-2}u_\mu(\tilde{u}_\mu - u_\mu)^+ \, dz \\
&= \int_\Omega \left[\mu\beta(z)\tilde{u}_\mu^{-\eta} + f(z,\tilde{u}_\mu) + \hat{\xi}_\rho \tilde{u}_\mu^{p-1}\right](\tilde{u}_\mu - u_\mu)^+ dz \text{ (see (77))} \\
&\geq \int_\Omega \mu\beta(z)\tilde{u}_\mu^{-\eta}(\tilde{u}_\mu - u_\mu)^+ \, dz \text{ (see hypothesis } H_2(\text{iv})) \\
&= \langle V(\tilde{u}_\mu), (\tilde{u}_\mu - u_\mu)^+\rangle + \int_\Omega \hat{\xi}_\rho \tilde{u}_\mu^{p-1}(\tilde{u}_\mu - u_\mu)^+ \, dz \\
&\quad \text{(since } \tilde{u}_\mu \text{is the unique solution of problem (71))},
\end{aligned}$$

$$\Rightarrow \tilde{u}_\mu \le u_\mu \text{ (see Proposition 2.3).} \tag{79}$$

Next in (78), we choose the test function $h = (u_\mu - u_\lambda)^+ \in W_0^{1,\vartheta}(\Omega)$. Then

$$\begin{aligned}
&\langle V(u_\mu), (u_\mu - u_\lambda)^+\rangle + \int_\Omega \hat{\xi}_\rho u_\mu^{p-1}(u_\mu - u_\lambda)^+ \, dz \\
&= \int_\Omega \left[\mu\beta(z)u_\lambda^{-\eta} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1}\right](u_\mu - u_\lambda)^+ \, dz \text{ (see (77))} \\
&\le \int_\Omega \left[\lambda\beta(z)u_\lambda^{-\eta} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1}\right](u_\mu - u_\lambda)^+ \, dz \text{ (since } \mu < \lambda) \\
&= \langle V(u_\lambda), (u_\mu - u_\lambda)^+\rangle + \int_\Omega \hat{\xi}_\rho u_\lambda^{p-1}(u_\mu - u_\lambda)^+ \, dz \text{ (since } u_\lambda \in S_\lambda^1),
\end{aligned}$$

$$\Rightarrow u_\mu \le u_\lambda \text{ (see Proposition 2.3).} \tag{80}$$

From (79) and (80), we see that

$$\tilde{u}_\mu \le u_\mu \le u_\lambda,$$

$$\Rightarrow u_\mu \in S_\mu^1 \text{ (see (77)) and so } \mu \in \mathcal{L}_1.$$

The proof is now complete. □

Hidden in the above proof is the following weak monotonicity property of the solution multifunction $\lambda \mapsto S_\lambda^1$.

Corollary 4.4. *If hypotheses H_0 and H_2 hold, $\lambda \in \mathcal{L}_1$, $u_\lambda \in S_\lambda^1$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}_1$ and there exists $u_\mu \in S_\mu^1$ such that $u_\mu \le u_\lambda$.*

From Proposition 4.3, we have that $\mathcal{L}_1$ is an interval. Next we show that this interval is bounded.

Let $\lambda^* = \sup \mathcal{L}_1$.

Proposition 4.5. *If hypotheses H_0 and H_2 hold, then $\lambda^* < \infty$.*

Proof. Hypothesis H_2(ii) implies that we can find $M_0 > 0$ such that

$$\beta(z)x^{p-1} \le f(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \ge M_0. \tag{81}$$

Choose $\tilde{\lambda} > 1$ large so that

$$\beta(z)M_0^{p-1} \le \tilde{\lambda}\beta(z)M_0^{-\eta} - \hat{c}_\delta(1 + M_0^{d-1}) \text{ (see hypothesis } H_2\text{(iii)).} \tag{82}$$

Then for a.a. $z \in \Omega$ and all $0 < x < M_0$, we have

$$\begin{aligned}
\beta(z)x^{p-1} &\le \beta(z)M_0^{p-1} \\
&\le \tilde{\lambda}\beta(z)M_0^{-\eta} - \hat{c}_\delta(1 + M_0^{d-1}) \\
&\le \tilde{\lambda}\beta(z)x^{-\eta} - \hat{c}_\delta(1 + x^{d-1}) \\
&\le \tilde{\lambda}\beta(z)x^{-\eta} + f(z, x) \text{ (see hypothesis } H_2\text{(iii)).}
\end{aligned} \tag{83}$$

From (81) and (83), it follows that

$$\beta(z)x^{p-1} \le \tilde{\lambda}\beta(z)x^{-\eta} + f(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{84}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}_1$. Then we can find $u_\lambda \in S_\lambda^1 \subseteq W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec u_\lambda$(see Proposition 4.2). Consider $\Omega_0 \subseteq \Omega$ open with $\bar{\Omega}_0 \subseteq \Omega$. Then since $0 \prec u_\lambda$, we have

$$0 < \hat{\mu}_0 = \mathrm{essinf}_{\Omega_0} u_\lambda.$$

Let $\epsilon > 0$ and set $\hat{\mu}_0^\epsilon = \hat{\mu}_0 + \epsilon$. Also for $\rho = \|u_\lambda\|_\infty$, let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv). We have

$$\begin{aligned} 0 &\leq \hat{\mu}_0^{-\eta} - (\hat{\mu}_0^\epsilon)^{-\eta} \\ &\leq \frac{(\mu_0^\epsilon)^\eta - \mu_0^\eta}{\hat{\mu}_0^{2\eta}} \\ &\leq \left[\frac{\epsilon}{\hat{\mu}_0^2}\right]^\eta . \end{aligned} \tag{85}$$

In Ω_0, we have

$$\begin{aligned} &-\Delta_p^a \hat{\mu}_0^\epsilon - \Delta_q \hat{\mu}_0^\epsilon + \hat{\xi}_\rho (\hat{\mu}_0^\epsilon)^{p-1} - \lambda\beta(z)(\hat{\mu}_0^\epsilon)^{-\eta} \\ &\leq \hat{\xi}_\rho \hat{\mu}_0^{p-1} - \lambda\beta(z)\hat{\mu}_0^{-\eta}\hat{\gamma}_\lambda(\epsilon) \text{ with } \hat{\gamma}_\lambda(\epsilon) \to 0^+ \text{ as } \epsilon \to 0^+ \text{ (see (85))} \\ &\leq \left[\hat{\xi}_\rho + \beta(z)\right] \hat{\mu}_0^{p-1} - \lambda\beta(z)\hat{\mu}_0^{-\eta} + \hat{\gamma}_\lambda(\epsilon) \\ &\leq \tilde{\lambda}\beta(z)\hat{\mu}_0^{-\eta} + f(z, \hat{u}_0) + \hat{\xi}_\rho \mu_0^{p-1} - \lambda\beta(z)\hat{\mu}_0^{-\eta} + \hat{\gamma}_\lambda(\epsilon) \text{ (see (84))} \\ &\leq f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} - [\lambda - \tilde{\lambda}]\beta(z)\hat{\mu}_0^{-\eta} + \hat{\gamma}_\lambda(\epsilon) \text{ (see hypothesis } \mathrm{H}_2\text{(iv))} \\ &\leq f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} - k \text{ for } \epsilon \in (0,1) \text{ small, with } k > 0 \text{ (recall that } \lambda > \tilde{\lambda}) \\ &= -\Delta_p^a u_\lambda - \Delta_q u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - k \text{ in } \Omega_0 \text{ (since } u_\lambda \in S_\lambda). \end{aligned}$$

Note that if $h = (\hat{\mu}_0^\epsilon - u_\lambda)^+ \in W^{1,\theta}(\Omega_0)$, then $0 \leq h(z) \leq \epsilon$ for a.a. $z \in \Omega_0$. Acting on the inequality with h and using the nonlinear Green's identity (see [28, p.34]), we deduce that for $\epsilon \in (0,1)$ even smaller if necessary, we have $\hat{\mu}_0^\epsilon \leq u_\lambda$. This contradicts the definition of $\hat{\mu}_0$. Therefore $\lambda \notin \mathcal{L}_1$ and so $\lambda^* \leq \tilde{\lambda} < \infty$. □

From Propositions 4.3 and 4.5, it follows that

$$(0, \lambda^*) \subseteq \mathcal{L}_1 \subseteq (0, \lambda^*].$$

Next we show that for $\lambda \in (0, \lambda^*)$, we have multiple solutions.

Proposition 4.6. *If hypotheses H_0 and H_2 hold and $0 < \lambda < \lambda^*$, then problem $(P_\lambda)^1$ has at least two solutions $u_0, \hat{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec u_0$, $0 \prec \hat{u}$.*

Proof. Let $0 < \mu < \lambda < \sigma < \lambda^*$. According to Corollary 4.4, we can find $u_\mu \in S_\mu^1$, $u_\sigma \in S_\sigma^1$ such that

$$u_\mu \leq u_\sigma. \tag{86}$$

Then for $\rho = \|u_\sigma\|_\infty$, let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv). We introduce the Carathéodory function $\hat{\eta}_\lambda(z,x)$ defined by (see (86))

$$\hat{\eta}_\lambda(z,x) = \begin{cases} \lambda\beta(z)u_\mu(z)^{-\eta} + f(z,u_\mu(z)) + \hat{\xi}_\rho u_\mu(z)^{p-1} & \text{if } x < u_\mu(z), \\ \lambda\beta(z)x^{-\eta} + f(z,x) + \hat{\xi}_\rho x^{p-1} & \text{if } u_\mu(z) \le x \le u_\sigma(z), \\ \lambda\beta(z)u_\sigma(z)^{-\eta} + f(z,u_\sigma(z)) + \hat{\xi}_\rho u_\sigma(z)^{p-1} & \text{if } u_\sigma(z) < x. \end{cases} \tag{87}$$

We set $\hat{N}_\lambda(z,x) = \int_0^x \hat{\eta}_\lambda(z,s)\,ds$ and consider the C^1-functional $\hat{j}_\lambda : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{j}_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q + \frac{\hat{\xi}_\rho}{p}\|u\|_p^p - \int_\Omega \hat{N}_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

Evidently, relation (82) and Proposition 2.2 imply that $\hat{j}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous (see Proposition 2.1). Therefore, we can find $u_0 \in W_0^{1,\vartheta}(\Omega)$ such that

$$\begin{aligned} \hat{j}_\lambda(u_0) &= \inf\left\{\hat{j}_\lambda(u) : u \in W_0^{1,\vartheta}(\Omega)\right\}, \\ &\Rightarrow \langle \hat{j}_\lambda'(u_0), h\rangle = 0 \text{ for all } h \in W_0^{1,\vartheta}(\Omega), \\ \Rightarrow \langle V(u_0), h\rangle + \int_\Omega \hat{\xi}_\rho |u_0|^{p-2}u_0 h\,dz &= \int_\Omega \hat{\eta}_\lambda(z,u_0)h\,dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega). \end{aligned} \tag{88}$$

In (88), first we choose the test function $h = (u_\mu - u_0)^+ \in W_0^{1,\vartheta}(\Omega)$ and obtain $u_\mu \le u_0$ and then use the test function $h = (u_0 - u_\sigma)^+ \in W_0^{1,\vartheta}(\Omega)$ and show that $u_0 \le u_\sigma$. Finally we have

$$u_\mu \le u_0 \le u_\sigma \Rightarrow u_0 \in S_\lambda^1 \text{ (see (87), (88))}.$$

Next, let $\eta_\lambda(z,x)$ be the function defined by

$$\eta_\lambda(z,x) = \begin{cases} \lambda\beta(z)u_\mu(z)^{-\eta} + f(z,u_\mu(z)) + \hat{\xi}_\rho u_\mu(z)^{p-1} & \text{if } x < u_\mu(z), \\ \lambda\beta(z)x^{-\eta} + f(z,x) + \hat{\xi}_\rho x^{p-1} & \text{if } u_\mu(z) \le x. \end{cases} \tag{89}$$

We set $N_\lambda(z,x) = \int_0^x \eta_\lambda(z,s)\,ds$ and consider the C^1-functional $j_\lambda : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$j_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q + \frac{\hat{\xi}_\rho}{p}\|u\|_p^p - \int_\Omega N_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

From Proposition 3.7, we know that

$$C_k(j_\lambda,u_0) = C_k(\hat{j}_\lambda,u_0) = \delta_{k,0}\mathbb{R} \text{ for all } k \in \mathbb{N}_0 (\text{since } u_0 \text{ is a global minimizer of } \hat{j}_\lambda(\cdot)),$$

$$\Rightarrow u_0 \text{ is a local minimizer of } j_\lambda(\cdot) \text{ (see [5],p.43)}. \tag{90}$$

Also, from Proposition 3.9, we know that

$$j_\lambda(\cdot) \text{ satisfies the C-condition.} \tag{91}$$

From (89) it follows that

$$\begin{aligned} &K_{j_\lambda} \subseteq [u_\mu) \cap L^\infty(\Omega), \\ &\Rightarrow K_{j_\lambda} \text{ is finite.} \end{aligned} \tag{92}$$

(otherwise we already have an infinity of bounded, positive solutions and so we are done). Then (90), (91), (92) and Theorem 5.7.6, p.449, of Papageorgiou-Rădulescu-Repovš [28], imply that we can find $\rho \in (0, 1)$ small such that

$$j_\lambda(u_0) < \inf\{j_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda. \tag{93}$$

If $u \in C_0^1(\bar{\Omega}), u(z) > 0$ for all $z \in \Omega$, then on account of hypothesis H_2(ii), we have

$$j_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{94}$$

Then (91), (93), (94) and the mountain pass theorem, imply that we can find $\hat{u} \in W_0^{1,\vartheta}(\Omega)$ such that

$$\hat{u} \in K_{j_\lambda} \subseteq [u_\mu) \cap L^\infty(\Omega),\ j_\lambda(u_0) < m_\lambda \le j_\lambda(\hat{u}),$$
$$\Rightarrow \hat{u} \in S_\lambda^1 \text{ (see (89))},\ \hat{u} \not\equiv u_0.$$

This completes the proof. □

It remains to decide about the admissibility of the critical parameter $\lambda^* > 0$.

Proposition 4.7. *If hypotheses H_0 and H_2 hold, then $\lambda^* \in \mathcal{L}_1$.*

Proof. Let $\lambda_n \to (\lambda^*)^-$ and $u_n \in S^1_{\lambda_n}, n \in \mathbb{N}$. From the proof of Proposition 4.6, we know that we can have

$$u_1 \le u_n \text{ and } j_{\lambda_n}(u_n) \le j_{\lambda_n}(u_1) \le j_{\lambda^*}(u_1) \le \hat{M} \text{ for some } \hat{M} > 0, \text{ all } n \in \mathbb{N}. \tag{95}$$

Also, we have

$$\langle V(u_n), h\rangle = \int_\Omega [\lambda_n \beta(z) u_n^{-\eta} + f(z, u_n)] h \, dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega). \tag{96}$$

Then using (95), (96) and reasoning as in the proof of Proposition 3.9(see the "**Claim**" in that proof), we obtain

$$u_n \to u^* \text{ in } W_0^{1,\vartheta}(\Omega), u_1 \le u^*,$$
$$\Rightarrow u^* \in S_\lambda^1 \text{ and so } \lambda^* \in \mathcal{L}_1.$$

The proof is now complete. □

Therefore, we have

$$\mathcal{L}_1 = (0, \lambda^*].$$

We can state the following existence and multiplicity theorem for problem $(P_\lambda)^1$. The result is global in the parameter $\lambda > 0$(a bifurcation-type result).

Theorem 4.8. *If hypotheses H_0 and H_2 hold, then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$ problem $(P_\lambda)^1$ has at least two solutions*

$$u_0, \hat{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0,\ 0 \prec \hat{u};$$

(b) *for $\lambda = \lambda^*$ problem $(P_\lambda)^1$ has at least one solution*

$$u^* \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u^*;$$

(c) *for all $\lambda > \lambda^*$ problem $(P_\lambda)^1$ has no solution.*

Remark 4.9. Hypotheses H_2 incorporate both cases where $f \geq 0$ and f is sign changing. So Theorem 4.8 generalizes all previous results, even for balanced growth problems. See the works of Papageorgiou-Rădulescu-Repovš [29](balanced growth problems) and of Liu-Papageorgiou [21], Han-Liu-Papageorgiou [14](double phase problems, in [21] $f(z,x) = \zeta(z)x^{r-1}$, with $\zeta \in L^\infty(\Omega)_+, p < r < q^*$ and in [14], $f(z,\cdot)$ is negative near zero).

5. Problem $(P_\lambda)^2$

In this section, we examine problem $(P_\lambda)^2$. This problem can be viewed as a singular perturbation of a nonlinear eigenvalue problem for the double phase operator.

As we did for problem $(P_\lambda)^1$, we introduce the following two sets:

$$\mathcal{L}_2 = \left\{\lambda > 0 : \text{problem } (P_\lambda)^2 \text{ has a solution}\right\},$$

$$S_\lambda^2 = \text{the solution set of } (P_\lambda)^2.$$

The hypotheses on the function $f(z,x)$ are the following:

(H_3) : $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is an $L^\infty(\Omega)$-locally Lipschitz function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \leq f(z,x) \leq \hat{a}(z)[1 + x^{r-1}]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\hat{a} \in L^\infty(\Omega)_+, p < r < q^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exist $\hat{c} > 0$ and $\tau \in [1,q)$ such that

$$-\hat{c}(1 + x^\tau) \leq f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0;$$

(iii) for every $s > 0$, there exists $m_s > 0$ such that

$$m_s \leq f(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq s;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0,\rho]$.

Remark 5.1. The hypotheses are similar to those in H_2, with one major difference. Now we have $f \geq 0$. Note that $f(z,\cdot)$ remains $(p-1)$-superlinear without satisfying the AR-condition. As before, we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$.

We start by considering the following purely singular problem

$$\left\{\begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \beta(z)u(z)^{-\eta}, \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{array}\right\} \tag{97}$$

We already know from Proposition 3.3, that the above problem has a unique solution

$$\bar{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega), \ 0 \prec \bar{u}.$$

In addition, from Proposition 3.4, we have that

$$\beta(\cdot)\bar{u}(\cdot)^{-\eta} \in L^\infty(\Omega).$$

Proposition 5.2. *If hypotheses H_0 and H_3 hold, then $\mathcal{L}_2 \neq \emptyset$ and for every $\lambda \in \mathcal{L}_2$, we have*

$$\emptyset \neq S^2_\lambda \subseteq W^{1,\vartheta}_0(\Omega) \cap L^\infty(\Omega),\ 0 \prec u \text{ for all } u \in S^2_\lambda.$$

Proof. Consider the following auxiliary Dirichlet problem

$$-\Delta^a_p u(z) - \Delta_q u(z) = \beta(z)\bar{u}^{-\eta} + 1 \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{98}$$

We rewrite (98) as the following equivalent abstract operator equation

$$V(u) = \beta\bar{u}^{-\eta} + 1 \text{ in } W^{1,\vartheta}_0(\Omega)^*.$$

The operator $V(\cdot)$ is maximal monotone, coercive, hence it is surjective. So, we can find $\tilde{u} \in W^{1,\vartheta}_0(\Omega)$ such that

$$V(\tilde{u}) = \beta\bar{u}^{-\eta} + 1 \text{ in } W^{1,\vartheta}_0(\Omega)^*.$$

Moreover, the strict monotonicity of $V(\cdot)$(see Proposition 2.3), implies that this solution is unique. From Proposition 2.6, we have

$$\tilde{u} \in W^{1,\vartheta}_0(\Omega) \cap L^\infty(\Omega).$$

So, problem (98) has a unique solution $\tilde{u}$. We have

$$\begin{aligned} -\Delta^a_p\bar{u} - \Delta_q\bar{u} &= \beta(z)\bar{u}^{-\eta} \\ &\leq \beta(z)\bar{u}^{-\eta} + 1 \\ &= -\Delta^a_p\tilde{u} - \Delta_q\tilde{u} \text{ in } \Omega, \end{aligned}$$

$$\Rightarrow \bar{u} \leq \tilde{u} \text{ (by the weak comparison principle, see [33])}. \tag{99}$$

On account of hypothesis H_3(i) and since $\tilde{u} \in L^\infty(\Omega)$, we see that

$$f(\cdot, \tilde{u}(\cdot)) \in L^\infty(\Omega).$$

So, we can find $\lambda_0 > 0$ such that

$$0 \leq \lambda f(z, \tilde{u}(z)) \leq 1 \text{ for a.a. } z \in \Omega,\ \text{all } 0 < \lambda \leq \lambda_0. \tag{100}$$

Then we obtain

$$\begin{aligned} -\Delta^a_p\tilde{u} - \Delta_q\tilde{u} &= \beta(z)\bar{u}^{-\eta}+ \\ &\geq \beta(z)\bar{u}^{-\eta} + \lambda f(z, \tilde{u}(z)) \text{ (see (100))} \\ &\geq \beta(z)\tilde{u}^{-\eta} + \lambda f(z, \tilde{u}(z)) \text{ in } \Omega \text{ (see (99))}. \end{aligned} \tag{101}$$

For $0 < \lambda \leq \lambda_0$, we introduce the Carathéodory function $\hat{t}_\lambda(z, x)$ defined by

$$\hat{t}_\lambda(z, x) = \begin{cases} \beta(z)\bar{u}(z)^{-\eta} + \lambda f(z, \bar{u}(z)) & \text{if } x < \bar{u}(z), \\ \beta(z)x^{-\eta} + \lambda f(z, x) & \text{if } \bar{u}_\mu(z) \leq x \leq \tilde{u}(z), \\ \beta(z)\tilde{u}(z)^{-\eta} + \lambda f(z, \tilde{u}(z)) & \text{if } \tilde{u}(z) < x. \end{cases} \tag{102}$$

We set $\hat{T}_\lambda(z, x) = \int_0^x \hat{t}_\lambda(z, s)\, ds$ and consider the C^1-functional $\hat{l}_\lambda$: $W^{1,\vartheta}_0(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{l}_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|^q_q - \int_\Omega \hat{T}_\lambda(z, u)\, dz \text{ for all } u \in W^{1,\vartheta}_0(\Omega).$$

From Proposition 2.2 and (101) it is clear that $\hat{l}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W_0^{1,\vartheta}(\Omega)$ such that

$$\hat{l}_\lambda(u_\lambda) = \inf\left\{\hat{l}_\lambda(u) : u \in W_0^{1,\vartheta}(\Omega)\right\},$$

$$\Rightarrow \langle \hat{l}'_\lambda(u_\lambda), h\rangle = 0 \text{ for all } h \in W_0^{1,\vartheta}(\Omega),$$

$$\Rightarrow \langle V(u_\lambda), h\rangle = \int_\Omega \hat{t}_\lambda(z, u_\lambda) h\, dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega). \tag{103}$$

In (103), we choose the test function $h = (\bar{u} - u_\lambda)^+ \in W_0^{1,\vartheta}(\Omega)$. Then since $0 \le f(z, \bar{u}(z))$ for a.a. $z \in \Omega$, we obtain

$$\begin{aligned} &\langle V(u_\lambda), (\bar{u} - u_\lambda)^+\rangle \\ &\ge \int_\Omega \beta(z)\bar{u}^{-\eta}(\bar{u} - u_\lambda)^+\, dz \text{ (see (102))} \\ &= \langle V(\bar{u}), (\bar{u} - u_\lambda)^+\rangle, \end{aligned}$$

$$\Rightarrow \bar{u} \le u_\lambda \text{ (see Proposition 2.3)}.$$

Also, if in (103) we choose the test function $h = (u_\lambda - \tilde{u})^+ \in W_0^{1,\vartheta}(\Omega)$ and use (101), we show that

$$u_\lambda \le \tilde{u}.$$

We have proved that

$$\begin{aligned} &\bar{u} \le u_\lambda \le \tilde{u}, \\ &\Rightarrow u_\lambda \in S^2_\lambda \text{(see (102), (103)) and so } (0, \lambda_0] \subseteq \mathcal{L}_2 \ne \emptyset. \end{aligned}$$

From Proposition (2.6), we have that

$$S^2_\lambda \subseteq W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega) \text{ for all } \lambda \in \mathcal{L}_2.$$

Moreover, if $u \in S^2_\lambda$, then

$$-\Delta_p^a u_\lambda - \Delta_q u_\lambda \ge 0 \text{ in } \Omega, \Rightarrow 0 \prec u \text{ (see [31])}.$$

The proof is now complete. □

As we did for $\mathcal{L}_1$, we show that $\mathcal{L}_2$ is connected (an interval).

Proposition 5.3. *If hypotheses H_0 and H_3 hold, $\lambda \in \mathcal{L}_2$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}_2$.*

Proof. Let $u_\lambda \in S^2_\lambda$. Since $f \ge 0$ as in the proof of Proposition 4.3, we show that

$$\bar{u} \le u_\lambda$$

with $\bar{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec \bar{u}$ being the unique solution of (97). Then we can introduce the Carathéodory function $\tilde{k}_\mu(z, x)$ defined by

$$\tilde{k}_\mu(z, x) = \begin{cases} \beta(z)\bar{u}(z)^{-\eta} + \mu f(z, \bar{u}(z)) & \text{if } x < \bar{u}(z), \\ \beta(z)x^{-\eta} + \mu f(z, x) & \text{if } \bar{u}(z) \le x \le u_\lambda(z), \\ \beta(z)u_\lambda(z)^{-\eta} + \mu f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{104}$$

We set $\tilde{K}_\mu(z,x) = \int_0^x \tilde{k}_\mu(z,s)\,ds$ and consider the C^1-functional $\tilde{e}_\mu : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\tilde{e}_\mu(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \tilde{K}_\mu(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

Applying the Weierstrass-Tonelli theorem, we find $u_\mu \in W_0^{1,\vartheta}(\Omega)$ such that

$$\tilde{e}_\mu(u_\mu) = \inf\left\{\tilde{e}_\mu(u) : u \in W_0^{1,\vartheta}(\Omega)\right\},$$
$$\Rightarrow \langle \tilde{e}_\mu'(u_\mu), h\rangle = 0 \text{ for all } h \in W_0^{1,\vartheta}(\Omega),$$
$$\Rightarrow \langle V(u_\mu), h\rangle = \int_\Omega \tilde{k}_\mu(z,u_\mu)h\,dz \text{ for all } h \in W_0^{1,\vartheta}(\Omega).$$

Choosing the test functions $h = (\bar{u} - u_\mu)^+ \in W_0^{1,\vartheta}(\Omega)$ and $h = (u_\mu - u_\lambda)^+ \in W_0^{1,\vartheta}(\Omega)$, we show that

$$\bar{u} \le u_\mu \le u_\lambda, \Rightarrow u_\mu \in S_\mu^2 \text{ (see (104)) and so } \mu \in \mathcal{L}_2,$$

which completes the proof. □

Again embedded in the above proof, is the following weak monotonicity property of the solution multifunction $\lambda \mapsto S_\lambda^2$.

Corollary 5.4. *If hypotheses H_0 and H_3 hold, $\lambda \in \mathcal{L}_2, u_\lambda \in S_\lambda^2$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}_2$ and there exists $u_\mu \in S_\mu^2$ such that $u_\mu \le u_\lambda$.*

So far , we have proved that $\mathcal{L}_2$ is an interval in $\mathring{\mathbb{R}}_+ = (0,+\infty)$. Next we show that this interval is bounded.

Let $\hat{\lambda}^* = \sup \mathcal{L}_2$.

Proposition 5.5. *If hypotheses H_0 and H_3 hold, then $\hat{\lambda}^* < \infty$.*

Proof. We reason as in the proof of Proposition 4.5. Hypothesis H_3(ii) implies that we can find $\tilde{M}_1 > 1$ such that

$$\|\beta\|_\infty x^{p-1} \le f(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \ge \tilde{M}_1. \tag{105}$$

Also note that

$$\beta(z)x^{p-1} \le \beta(z)x^{-\eta} \text{ for a.a. } z \in \Omega, \text{ all } 0 < x \le 1. \tag{106}$$

Let $m_1 > 0$ be as postulated by hypothesis H_3(iii). Then for $\tilde{\lambda} > 1$ large, we will have

$$\beta(z)x^{p-1} \le \beta(z)\tilde{M}_1^{p-1} \le \beta(z)\tilde{M}_1^{-\eta} + \tilde{\lambda}m_1 \le \beta(z)x^{-\eta} + \tilde{\lambda}m_1 \text{ for a.a. } z \in \Omega, \text{ all } 1 < x < \tilde{M}_1. \tag{107}$$

From (105), (106), (107), we infer that

$$\beta(z)x^{p-1} \le \beta(z)x^{-\eta} + \tilde{\lambda}f(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{108}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}_2$. We can find $u_\lambda \in S_\lambda^2 \subseteq W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec u_\lambda$. Using u_λ, (108) and proceeding as in the proof of Proposition 4.5, we reach a contradiction. Hence $\lambda \notin \mathcal{L}_2$ and so $0 < \hat{\lambda}^* \le \tilde{\lambda}$. □

It follows that

$$(0, \hat{\lambda}^*) \subseteq \mathcal{L}_2 \subseteq (0, \hat{\lambda}^*]. \tag{109}$$

Proposition 5.6. *If hypotheses H_0 and H_3 hold and $0 < \lambda < \hat{\lambda}^*$, then problem $(P_\lambda)^2$ has at least two solutions*

$$u_0, \hat{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0,\ 0 \prec \hat{u}.$$

Proof. Let $0 < \mu < \lambda < l < \hat{\lambda}^*$. We know that $\mu, l \in \mathcal{L}_2$(see (109)) and so using Corollary 5.4, we can find $u_\mu \in S^2_\mu, u_l \in S^2_l$ with $u_\mu \leq u_l$. So, we can introduce the Carathéodory function $\hat{\gamma}_\lambda(z, x)$ defined by

$$\hat{\gamma}_\lambda(z,x) = \begin{cases} \beta(z)u_\mu(z)^{-\eta} + \lambda f(z, u_\mu(z)) & \text{if } x < u_\mu(z), \\ \beta(z)x^{-\eta} + \lambda f(z,x) & \text{if } u_\mu(z) \leq x \leq u_l(z), \\ \beta(z)u_l(z)^{-\eta} + \lambda f(z, u_l(z)) & \text{if } u_l(z) < x. \end{cases} \tag{110}$$

We set $\hat{\Gamma}_\lambda(z,x) = \int_0^x \hat{\gamma}_\lambda(z,s)\,ds$ and consider the C^1-functional $\hat{b}_\lambda : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{b}_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{\Gamma}_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

We can find $u_0 \in W_0^{1,\vartheta}(\Omega)$ such that

$$\hat{b}_\lambda(u_0) = \inf\left\{\hat{b}_\lambda(u) : u \in W_0^{1,\vartheta}(\Omega)\right\}, \tag{111}$$

$$\Rightarrow u_0 \in K_{\hat{b}_\lambda} \text{ and so } u_\mu \leq u_0 \leq u_l \text{ (see (110))},$$

$$\Rightarrow u_0 \in S^2_\lambda.$$

Also, we introduce the Carathéodory function $\gamma_\lambda(z, x)$ defined by

$$\gamma_\lambda(z,x) = \begin{cases} \beta(z)u_\mu(z)^{-\eta} + \lambda f(z, u_\mu(z)) & \text{if } x < u_\mu(z), \\ \beta(z)x^{-\eta} + \lambda f(z,x) & \text{if } u_\mu(z) \leq x. \end{cases} \tag{112}$$

We set $\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)\,ds$ and consider the C^1-functional $b_\lambda : W_0^{1,\vartheta}(\Omega) \mapsto \mathbb{R}$ defined by

$$b_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \Gamma_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

We have

$$C_k(b_\lambda, u_0) = C_k(\hat{b}_\lambda, u_0) = \delta_{k,0}\mathbb{R} \text{ for all } k \in \mathbb{N}_0 \text{ (see (111) and Proposition 3.7)},$$

$$\Rightarrow u_0 \text{ is a local minimizer of } b_\lambda(\cdot) \text{ (see [5])}. \tag{113}$$

Using (112), we show that

$$K_{b_\lambda} \subseteq [u_\mu) \cap L^\infty(\Omega). \tag{114}$$

On account of (114) and (112), we see that we may assume that K_{b_λ} is finite (otherwise we already have an infinity of bounded solutions for $(P_\lambda)^2$). Note that

$$b_\lambda(\cdot) \text{ satisfies the C-condition (see Proposition 3.9)}. \tag{115}$$

Therefore, we can find $\rho \in (0,1)$ small such that

$$b_\lambda(u_0) = \inf\{b_\lambda(u) : \|u-u_0\| = \rho\} = m_\lambda (\text{see } (113) \text{ and Theorem 5.7.6, p.449, of [28]}). \tag{116}$$

If $u \in C_0^1(\bar{\Omega})$ with $u(z) > 0$ for all $z \in \Omega$, then hypothesis H_3(iii) implies that

$$b_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{117}$$

Then (115), (116), (117) and the mountain pass theorem, imply that there exist $\hat{u} \in W_0^{1,\vartheta}(\Omega)$ such that

$$\hat{u} \in K_{b_\lambda} \subseteq [u_\mu) \cap L^\infty(\Omega),\ m_\lambda \le b_\lambda(\hat{u}),$$

$$\Rightarrow \hat{u} \in S_\lambda^2,\ \hat{u} \ne u_0,\ 0 \prec \hat{u}.$$

The proof is now complete. □

Finally, arguing as in the proof of Proposition 4.7, we show the admissibility of $\hat{\lambda}^*$.

Proposition 5.7. *If hypotheses H_0 and H_3 hold, then $\hat{\lambda}^* \in \mathcal{L}_2$.*

Therefore, we conclude that

$$\mathcal{L}_2 = (0, \hat{\lambda}^*] \text{ (see (109))}.$$

Summarizing, we can state the following existence and multiplicity theorem for problem $(P_\lambda)^2$ which is global in $\lambda > 0$(a bifurcation theorem).

Theorem 5.8. *If hypotheses H_0 and H_3 hold, then there exists $\hat{\lambda}^* > 0$ such that*

(a) *for all $\lambda \in (0, \hat{\lambda}^*)$, problem $(P_\lambda)^2$ has at least two solutions $u_0, \hat{u} \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec u_0$, $0 \prec \hat{u}$;*

(b) *for $\lambda = \hat{\lambda}^*$, problem $(P_\lambda)^2$ has at least one solution*

$$u^* \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u^*;$$

(c) *for all $\lambda > \hat{\lambda}^*$, problem $(P_\lambda)^2$ has no solution.*

Remark 5.9. Theorem 5.8, extends the multiplicity result of Liu-Dai-Papageorgiou-Winkert [20], where $f(z,x) = x^{r-1}$ for all $z \in \Omega$, all $x \ge 0$ with $p < r < q^*$ and their result is not global in $\lambda > 0$(see [20], Theorem 1.1). Their approach is based on the Nehari method. If we try to drop the hypothesis $f \ge 0$, then we encounter serious difficulties which we were not able to overcome. It is clear from the proof of Proposition 5.2 that for $\lambda > 0$ small, we have at least two bounded solutions. But whether we can have a multiplicity theorem which is global in $\lambda > 0$ for $(P_\lambda)^2$ without the hypothesis $f \ge 0$, is an interesting open problem.

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