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POSITIVE BOUND STATES OF FRACTIONAL CHOQUARD EQUATIONS WITH UPPER HARDY–LITTLEWOOD–SOBOLEV CRITICAL EXPONENT*

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Abstract. We are interested in the existence of positive bound solutions for the following fractional Choquard equation: $\begin{cases} (-\Delta)^s u + V(x)u = (\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} dy)|u|^{2^*_{\mu,s}-2}u, x \in \Omega, \\ u = 0, x \in \mathbb{R}^N \backslash \Omega, \end{cases}$ where $\Omega \subset \mathbb{R}^N$ is an unbounded exterior domain, $\partial\Omega \neq \emptyset$, $\mathbb{R}^N \backslash \Omega$ is bounded, $s \in (0,1)$, $N > 2s$, $0 < \mu < \min\{N, 4s\}$, $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ is the fractional upper Hardy–Littlewood–Sobolev critical exponent, and $V \in L^{\frac{N}{2s}}(\Omega)$ is a nonnegative function. By combining variational methods and the Brouwer degree theory, we investigate the existence of positive bound solutions to this equation when $V(x)$ and the hole $\mathbb{R}^N \backslash \Omega$ are suitably small in some sense. The results obtained in this paper extend and improve on some recent works. Our result also holds true in the case $\Omega = \mathbb{R}^N$; hence, this paper can be viewed as an extension of recent contributions on the Benci–Cerami problem for the fractional Choquard equation.

Key words. bound state, fractional Choquard equation, Berestycki–Lions condition, Brouwer degree, upper Hardy–Littlewood–Sobolev critical exponent

MSC codes. 35R11, 35B33, 35B38, 47H11, 58E30, 81Q05

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1. Introduction and main results. In this article, we are interested in the following fractional Choquard equation:

$$
\begin{cases} (-\Delta)^s u + V(x)u = \Big(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} dy \Big) |u|^{2^*_{\mu,s}-2}u, x \in \Omega, \\ u = 0, x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded exterior domain, $\partial\Omega \neq \emptyset$, $\mathbb{R}^N \backslash \Omega$ is bounded, $s \in (0,1)$, $N > 2s$, $0 < \mu < \min\{N, 4s\}$, $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ is the fractional upper

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Hardy–Littlewood–Sobolev critical exponent, and $(-\Delta)^s$ is the fractional Laplace operator. The fractional Laplace operator was first introduced in the pioneering work of Laskin [26, 27]; for more details about the fractional Laplacian and fractional Sobolev spaces, we refer the interested reader to the monograph [39].

On the one hand, (1.1) stems from the following Choquard equation or nonlinear Schrödinger–Newton equation:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, x \in \mathbb{R}^3. \tag{1.2}$$

In the framework of quantum mechanics, (1.2) was elaborated by Pekar [46] in 1954. In the approximation to the Hartree–Fock theory of one-component plasma, Choquard used (1.2) to describe an electron trapped in its own hole. As an approximation of the Hartree–Fock theory, Bongers also investigated (1.2) in [6]. It is remarked that as a model of self-gravitating matter known in that context as the Schrödinger–Newton equation, this equation was studied by Penrose [47, 48]. To the best of our knowledge, Lieb [33] and Lions [36] first studied the existence and symmetry of positive solutions to (1.2). Since then, many authors have paid much attention to studying the existence, multiplicity, and properties of the solutions of the nonlinear Choquard equations, and indeed, many interesting results were obtained in the past decades. By using the rearrangements technique, the existence and uniqueness, up to translations, were investigated by Lieb and Loss in [35] and Lions in [37]. Furthermore, they proved the existence of a sequence of radially symmetric solutions by variational methods. In [52], Wei and Winter first proved the nondegeneracy and uniqueness of the ground state, and then they succeeded to obtain the multibump solutions for (1.2). Classification of solutions of (1.2) was first studied by Ma and Zhao [38]. More recently, Moroz and Van Schaftingen [42] completely studied the qualitative properties of solutions for the following Choquard equation:

$$-\Delta u + u = (K_\alpha * |u|^p)|u|^{p-2}u, x \in \mathbb{R}^N, \tag{1.3}$$

where $p > 1$, $N \in \{1, 2, \ldots\}$, and K_α is a Riesz potential defined by

$$K_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha |x|^{N-\alpha}} := \frac{\mathcal{C}_\alpha}{|x|^{N-\alpha}}.$$

Subsequently, Moroz and Van Schaftingen [43] gave a broad survey about Choquard equations. Especially, Gao et al. [15] and Guo et al. [17] independently studied positive high-energy solutions for the Benci–Cerami problem of the Choquard equation

$$-\Delta u + V(x)u = (K_\alpha * |u|^{2^*_\mu})|u|^{2^*_\mu - 2}u, u \in D^{1,2}(\mathbb{R}^N) \tag{1.4}$$

when $|V|_{\frac{N}{2}}$ is suitably small, where $0 < \mu < N$ if $N = 3$ or $N = 4$, $N - 4 \le \mu < N$ if $N \ge 5$, and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the upper Hardy–Littlewood–Sobolev critical exponent. Recently, Alves, Figueiredo, and Molle [2] considered the Choquard equation (1.4) with $V(x) = \lambda + V_0(x)$ and $\lambda \ge 0, V_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $0 < \mu < \min\{N, 4\}$, and $N \ge 3$ when V_0 and λ are suitably small; they obtained the existence of two positive solutions to (1.4). In fact, the results obtained in [2, 15, 17] extended the classical results due to Benci and Cerami [5] for the Schrödinger equation to the Choquard equation.

Compared with classical Choquard equations, studying the existence and multiplicity of solutions for fractional Choquard equations has not been addressed much in the literature. Especially, in the following, some articles related to our topic must be cited here. In [44], Mukherjee and Sreenadh studied the existence of weak solutions of the following doubly nonlocal fractional elliptic problem:

$$\begin{cases} (-\Delta)^s u = \left(\displaystyle\int_\Omega \frac{|u|^{2^*_{\mu,s}}}{|x-y|^\mu} dy\right)|u|^{2^*_{\mu,s}-2}u + \lambda u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, λ is a real parameter, $0 < \mu < N$, and $N > 2s$. They obtained some existence, nonexistence, and regularity results for the weak solution of the above problem using variational methods. In [19], He and Rădulescu were concerned with the qualitative analysis of positive solutions to the fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + V(x)u = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, x \in \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N),\ u(x) > 0,\ x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $s \in (0,1)$, $2s < N$, $0 < \alpha < \min\{N, 4s\}$, $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$, I_α is a Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N-\alpha}{2})\pi^{N/2}2^{N-\alpha}|x|^\alpha} := \frac{A_\alpha}{|x|^\alpha}, \tag{1.7}$$

and $V(x)$ satisfies the following conditions:

(i) The function V is positive on a set of positive measure.

(ii) $V \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$, where $1 < p_1 < \frac{2N-\alpha}{4s-\alpha} < p_2$ with $p_2 < \frac{N}{4s-N}$ if $2s < N < 4s$.

(iii) $|V|_{\frac{N}{2s}} < (2^{\frac{4s-\alpha}{2N-\alpha}} - 1)S_s^{\frac{(2s-N)[(N-\alpha)(1-s)+2s]+(2N-\alpha)2s}{2s(N-\alpha+2s)}}$, where S_s is the best Sobolev constant for the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$. By proving a version of the global compactness result of Struwe [50] for the case of fractional operators in $\mathbb{R}^N$, they showed that (1.6) has at least one bound state solution. Subsequently, He, Zhao, and Zou [20] also studied positive solutions to the fractional Choquard equation (1.6) under the following conditions:

(iv) $V \geq 0, \not\equiv 0$, and $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$.

(v) $|V|_{\frac{N}{2s}} < (2^{\frac{(4s-\alpha)^2}{(2N-\alpha)(N+2s-\alpha)}} - 1)S_s$.

It is noticed that the results obtained in [19, 20] are strongly dependent on the condition $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, which means that $V(x)$ may vanish at infinity. In [16], Guan, Rădulescu, and Wang obtained multiple bound state solutions for the fractional Choquard equation (1.6) when $V(x)$ is a positive potential bounded from below. In fact, the results obtained in [16] extended and improved on some works in [19, 20] in the case where the coefficient $V(x)$ vanishes at infinity.

On the other hand, (1.1) is closely related to the following classic local problems in the exterior domain:

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, x \in D, \\ u = 0, x \in \partial D, \end{cases} \tag{1.8}$$

where $D \subset \mathbb{R}^N (N \geq 3)$ is an unbounded domain, $\partial D \neq \emptyset$ is bounded, and $2 < p < \frac{2N}{N-2}$. In a classical paper by Benci and Cerami [4], the authors showed that (1.8) does not have any ground state solution. So they only find a bound state solution. For their purpose, the authors first analyzed the behavior of Palais–Smale sequences and proved a precise estimate of the energy levels where the Palais–Smale condition fails and then, using the variational method and the Brouwer degree theory, succeeded to obtain that the problem (1.8) has at least one positive solution for λ sufficiently small or for $\mathbb{R}^N\backslash D$ small enough. After this pioneering work, many local problems involving exterior domains were considered; we refer the reader to [3, 7, 8, 9, 11, 21, 25, 31, 32, 40, 41] and the references therein.

In recent years, some scholars have begun to extend the classic results obtained in [4] to some nonlocal problems. Specifically, Alves et al. [1] and Correia and Figueiredo [12] independently extended results obtained in [4] to the following fractional elliptic problems in the exterior domain:

$$\begin{cases} (-\Delta)^s u + u = |u|^{p-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N\backslash D, \end{cases} \tag{1.9}$$

where $D \subset \mathbb{R}^N$ is an exterior domain with a smooth boundary such that $\mathbb{R}^N\backslash D$ is bounded, $s \in (0,1)$, $N > 2s$, $p \in (2, 2_s^*)$, and $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent. In [1, 12], the authors first proved a version of the fractional operator in the unbounded domain of the global compactness result due to Struwe (see [50, 53]), then, combining with the Brouwer degree theory, barycentric functions, and the minimax argument, they obtained the existence of positive solutions for (1.9) provided that $\mathbb{R}^N\backslash D$ is contained in a small ball. Subsequently, Correia and Oliveira [13] investigated a positive solution for a class of fractional elliptic problems in exterior domains with small critical perturbations. In [10], Chen and Liu extended the classic results of [4] to the Kirchhoff-type nonlocal problem with $N = 3$. Soon afterward, Wang, Yuan, and Zhang [51] generalized the Kirchhoff-type nonlocal problem with subcritical nonlinearity, discussed in [10], to the Kirchhoff-type nonlocal problem with small critical perturbations. In [23], Jia, Li, and Ma studied the existence of positive solutions for a class of Kirchhoff-type problems in exterior domains with general nonlinear terms. Very recently, by establishing global compact lemmas, combining the variational method with the Brouwer degree theory, Jia, Li, and Ma [24] obtained a positive solution for a class of Kirchhoff-type nonlocal problems with critical exponents and nonconstant potential functions in exterior domains when the hole is suitably small. In [29], Ledesma and Miyagaki were concerned with the existence of positive solutions for the following fractional Choquard equation:

$$\begin{cases} (-\Delta)^s u + u = \Big(\displaystyle\int_D \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\Big)|u|^{p-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N\backslash D, \end{cases} \tag{1.10}$$

where $N > 2s$, $s \in (0,1)$, $0 < \alpha < N$, $D \subset \mathbb{R}^N$ is an unbounded exterior domain with smooth boundary $\partial D \neq \emptyset$, and $p \in (2, 2_s^*)$. First, to overcome the loss of uniqueness, as in [49], the authors investigated limit profiles of ground states of the limit problem of (1.10) as α sufficiently close to 0. Then, combined with the splitting lemma due to Struwe (see [50, 53]) and arguments used by Benci and Cerami [4], they succeeded to obtain a positive solution for (1.10) when $\mathbb{R}^N\backslash D$ small enough. In [30], Ledesma studied the existence of a positive solution for the fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + u = |u|^{q-2}u + \varepsilon\left(\displaystyle\int_D \frac{|u(y)|^p}{|x-y|^{\alpha}} dy\right)|u|^{p-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N\backslash D, \end{cases} \tag{1.11}$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $N > 2s$, $0 < \alpha < N$, $D \subset \mathbb{R}^N$ is an unbounded exterior domain with smooth boundary $\partial D \neq \emptyset$, $q \in (2, 2_s^*)$, and $p \in (\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2s})$. In fact, (1.11) could be viewed as an extension of fractional elliptic problems (1.9) with small Choquard-type nonlocal perturbations. Recently, Ye, Yu, and Tang [55] investigated the existence of positive solutions to the following fractional Choquard equation:

$$\begin{cases} (-\Delta)^s u + u = \left(\displaystyle\int_D \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\right) |u|^{p-2} u + \varepsilon \left(\displaystyle\int_D \frac{|u(y)|^{2^*_{\alpha,s}}}{|x-y|^{N-\alpha}} dy\right) |u|^{2^*_{\alpha,s}-2} u, x \in D, \\ u = 0, x \in \mathbb{R}^N \backslash D, \end{cases} \tag{1.12}$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $N > 2s$, $0 < \alpha < N$, $D \subset \mathbb{R}^N$ is an unbounded exterior domain with smooth boundary $\partial D \neq \emptyset$, $p \in (2, 2^*_{\alpha,s})$, and $2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}$. The authors first obtained the limit profiles and uniqueness of positive radial ground states for the limit equation of problems (1.12) without small critical perturbations as $\alpha \to N$. Then, using the variational method and the Brouwer degree theory, they obtained the existence of positive bound state solutions for (1.12) in case $\varepsilon > 0$ is sufficiently small.

Motivated by the above works, especially by [1, 4, 5, 12, 16, 19, 20, 24], in this paper, we are interested in the existence of positive bound state solutions to the Choquard equation (1.1) with nonconstant potential functions in an exterior domain.

The main result in the case of small perturbations from infinity of the indefinite potential establishes the following existence property of bound states.

THEOREM 1.1. *Suppose that V satisfies the following conditions:*

(V_1) $V \in L^{\frac{N}{2s}}(\Omega), V(x) \geq 0$ *and* $V(x) \not\equiv 0, x \in \Omega$.

(V_2) $0 < |V|_{\frac{N}{2s}} < (2^{\frac{4s-\mu}{2N-\mu}} - 1) S_s$.

Then there is a small $\lambda > 0$ such that if $\mathbb{R}^N \backslash \Omega \subset B_\lambda(0)$, then (1.1) *has at least one positive bound state solution.*

Remark 1.1. If V is a constant, then it is obvious that $V \notin L^{\frac{N}{2s}}(\mathbb{R}^N)$. However, the results about the fractional Choquard equation in exterior domains obtained in [29, 30] are strongly dependent on V being a constant potential function. So the methods used in [29, 30] seem to be not valid for our case.

Remark 1.2. To the best of our knowledge, when discussing critical problems (local problems or nonlocal problems) in exterior domains, the critical term is basically used as a small critical perturbation, except in [24]. Inspired by [4, 5, 24], in the case of small perturbations from infinity of the indefinite potential and nonsmall critical perturbations, we obtain the existence of positive bound states of the fractional Choquard equation (1.1) in exterior domains. Since there are double nonlocal characteristics in our equation which come from the nonlocal operator $(-\Delta)^s$ and the fractional Choquard nonlinear term, some refined estimates for our problem are very necessary and delicate.

Remark 1.3. It is particularly worth noting that our result also holds true in the case $\Omega = \mathbb{R}^N$ and hence can be viewed as an extension of recent results for the Benci-Cerami problem for the fractional Choquard equation with critical exponent (for specific details, see [19, 20]). Furthermore, since it is known but not completely trivial that $(-\Delta)^s$ reduces to the standard Laplacian $-\Delta$ as $s \to 1^-$, our results also extend the results obtained in [15, 17].

2. Preliminary results. Without any loss of generality, we may assume that $0\in\mathbb{R}^N\backslash\Omega$. As usual, for any $s\in(0,1)$, let

$$H^{s,2}(\mathbb{R}^N)=\Big\{u\in L^2(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx<\infty\Big\},$$

where the inner product and norm are defined by

$$(u,v)_{H^s}=\int_{\mathbb{R}^N}u(x)v(x)dx+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx,\ \|u\|^2_{H^s}=(u,u)_{H^s}.$$

The norms of u in $L^r(\Omega)$ and $L^r(\mathbb{R}^N)$ are denoted by $|u|_r$ and $|u|_{r,\mathbb{R}^N}$, $1\le r<\infty$. For any $s\in(0,1)$, define

$$D^{s,2}(\mathbb{R}^N)=\Big\{u\in L^{2^*_s}(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx<\infty\Big\}$$

with the Gagliardo seminorm

$$\|u\|^2_{\mathbb{R}^N}=(u,u)_{\mathbb{R}^N}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx.$$

According to Propositions 3.4 and 3.6 of [45], by omitting the normalization constant, we have

$$\|u\|^2_{\mathbb{R}^N}=|(-\Delta)^{\frac{s}{2}}u|^2_{2,\mathbb{R}^N}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx.$$

Then set

$$D^{s,2}_0(\Omega)=\Big\{u\in D^{s,2}(\mathbb{R}^N):u=0\text{ a.e. in }\mathbb{R}^N\backslash\Omega\Big\}$$

with the norm

$$\|u\|^2=(u,u)=\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx,$$

where $\mathcal{Q}:=(\mathbb{R}^N\times\mathbb{R}^N)\backslash(\Omega^c\times\Omega^c)$, $\Omega^c=\mathbb{R}^N\backslash\Omega$. According to the definition of $D^{s,2}_0(\Omega)$, it is obvious that $D^{s,2}_0(\Omega)\subset D^{s,2}(\mathbb{R}^N)$.

PROPOSITION 2.1 (see [34, 35]). *Let $t,r>1$ and $0<\mu<N$ with $1/t+\mu/N+1/r=2$, $f\in L^t(\mathbb{R}^N)$, and $h\in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,N,\mu,r)$ independent of f,h such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{\mu}}dx\,dy\Big|\le C(t,N,\mu,r)|f|_t\cdot|g|_r. \tag{2.1}$$

If $t=r=\frac{2N}{2N-\mu}$, then

$$C(t,N,\mu,r)=C(N,\mu)=\pi^{\frac{\mu}{2}}\frac{\Gamma(\frac{\pi-\mu}{2})}{\Gamma(\frac{2N-\mu}{2})}\Big(\frac{\Gamma(\frac{\pi}{2})}{\Gamma(N)}\Big)^{-1+\frac{\mu}{N}}.$$

In this case, the equality in (2.1) is achieved if and only if $f\equiv(const.)g$ and

$$g(x)=A(\gamma^2+|x-a|^2)^{-\frac{2N-\mu}{2}}$$

for some $A\in\mathbb{C}$, $0\neq\gamma\in\mathbb{R}$, and $a\in\mathbb{R}^N$.

LEMMA 2.1 (see [53]). *If $N>2s$ and $a\in L^{\frac{N}{2s}}(\mathbb{R}^N)$, then $\psi: D^{s,2}(\mathbb{R}^N)\to\mathbb{R}, u\mapsto \int_{\mathbb{R}^N} a(x)u^2\,dx$ is weakly continuous.*

Let $f=g=|u|^q$. Then, according to Proposition 2.1, we conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)u(y)|^q}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x$$

is well defined if $|u|^q\in L^t(\mathbb{R})$ for some $t>1$ with $\frac{2}{t}+\frac{\mu}{N}=2$. Therefore, for $u\in D^{s,2}(\mathbb{R}^N)$, thanks to Sobolev embedding theorems, we have

$$\frac{2N-\mu}{N}\le q\le\frac{2N-\mu}{N-2s}. \tag{2.2}$$

Hence, for any $u\in D^{s,2}(\mathbb{R}^N)$, we get

$$\left(\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\mu,s}}}\le(\mathcal{C}(N,\mu))^{\frac{1}{2^*_{\mu,s}}}|u|^2_{2^*_{\mu,s}},$$

where $\mathcal{C}(N,\mu):=A_\mu C(N,\mu)$.

From the above arguments, the energy functional associated with (1.1) is defined by

$$\begin{aligned}\mathcal{J}(u)&=\frac{1}{2}\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x+\frac{1}{2}\int_\Omega V(x)u^2\mathrm{d}x\\&\quad-\frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\&=\frac{1}{2}\|u\|^2+\frac{1}{2}\int_\Omega V(x)u^2\mathrm{d}x-\frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x,\ u\in D^{s,2}_0(\Omega).\end{aligned}$$

Furthermore, $\mathcal{J}(u)\in C^1(D^{s,2}_0(\Omega),\mathbb{R})$ and

$$\langle\mathcal{J}'(u),v\rangle=(u,v)+\int_\Omega V(x)uv\mathrm{d}x-\int_\Omega\int_\Omega\frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}-2}u(x)v(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x$$

for $u,v\in D^{s,2}_0(\Omega)$.

Define the Nehari manifold as

$$\mathcal{N}:=\{u\in D^{s,2}_0(\Omega))\backslash\{0\}:\mathcal{G}(u)=0\},\text{ where }\mathcal{G}(u):=\langle\mathcal{J}'(u),u\rangle.$$

Moreover, we have the following results about $\mathcal{N}$.

LEMMA 2.2. *Suppose that (V_1) holds. Then we have that*

(a) *$\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the unit sphere of $D^{s,2}_0(\Omega)$;*

(b) *$\mathcal{J}$ has a positive bound from below on $\mathcal{N}$;*

(c) *u is a critical point of $\mathcal{J}$ if and only if u is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$.*

Proof. Choose $u\in D^{s,2}_0(\Omega)$ with $\|u\|=1$, and define $f_u(t)$ by

$$f_u(t)=\frac{t^2}{2}\|u\|^2+\frac{t^2}{2}\int_\Omega V(x)u^2\mathrm{d}x-\frac{t^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x, t>0.$$

Obviously, $f_u(0)$ and $f_u'(0)=0$. Thanks to $V(x)\geq 0$, we have that

$$f_u(t)>0 \text{ for } t>0 \text{ small enough and } f_u(t)<0 \text{ for } t>0 \text{ large enough.}$$

Hence, there exists $t_u>0$ such that $f_u(t_u)=\max_{t\geq 0} f_u(t)$ with $f_u'(t_u)=0$ and $t_u u\in\mathcal{N}$. It is easy to see that t_u is unique. Furthermore, we have $f_u'(t)>0$ for $0<t<t_u$ and $f_u'(t)<0$ for $t>t_u$.

Since $\mathcal{J}\in C^2(D_0^{s,2}(\Omega),\mathbb{R})$, $\mathcal{G}$ is a C^1 functional. For any $u\in\mathcal{N}$, we have

$$\begin{aligned}\langle \mathcal{G}'(u),v\rangle &= 2\|u\|^2+2\int_\Omega V(x)u^2\mathrm{d}x-2\cdot 2^*_{\mu,s}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\ &=(2-2\cdot 2^*_{\mu,s})\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x<0,\end{aligned} \tag{2.3}$$

which implies that (a) holds.

For any $u\in\mathcal{N}$, since $V(x)\geq 0$, using the Sobolev inequality, there is $C_0>0$ such that

$$0=\|u\|^2+\int_\Omega V(x)u^2\mathrm{d}x-\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\geq \|u\|^2-C_0\|u\|^{2\cdot 2^*_{\mu,s}}.$$

Then there is $C_1>0$ such that

$$\|u\|\geq C_1 \text{ for any } u\in\mathcal{N}. \tag{2.4}$$

Consequently, by (2.4), we conclude that

$$\begin{aligned}\mathcal{J}(u)&=\frac{1}{2}\|u\|^2+\frac{1}{2}\int_\Omega V(x)u^2\mathrm{d}x-\frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\ &=\left(\frac{1}{2}-\frac{1}{2\cdot 2^*_{\mu,s}}\right)\|u\|^2>\left(\frac{1}{2}-\frac{1}{2\cdot 2^*_{\mu,s}}\right)C_1 \text{ for any } u\in\mathcal{N},\end{aligned}$$

which shows that (b) is satisfied.

If u is a critical point of $\mathcal{J}$ with $u\not\equiv 0$, then $\mathcal{J}'(u)=0$, and thus $\mathcal{G}(u)=0$. So u is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$. If u is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$, then there is $\varsigma\in\mathbb{R}$ satisfying $\mathcal{J}'(u)=\varsigma\mathcal{G}'(u)$. By $u\in\mathcal{N}$, we have

$$\langle\varsigma\mathcal{G}'(u),u\rangle=\langle\mathcal{J}'(u),u\rangle=0.$$

Hence, due to (2.3), we get $\varsigma=0$. That is, $\mathcal{J}'(u)=0$. ∎

Let $S_{\mu,s}$ be the best constant

$$S_{\mu,s}:=\inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^2_{\mathbb{R}^N}}{\left(\int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\mu,s}}}},$$

where I_μ is defined as in (1.7), and let S_s be the best Sobolev constant for the embedding $D^{s,2}(\mathbb{R}^N)\hookrightarrow L^{2^*_s}(\mathbb{R}^N)$, that is,

$$S_s=\inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^2_{\mathbb{R}^N}}{|u|^2_{2^*_s,\mathbb{R}^N}}.$$

It is well known that $S_{\mu,s}$ and S_s are both achieved at u if and only if

$$u(x) = C\left(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|}\right)^{\frac{N-2s}{2}}, x \in \mathbb{R}^N,$$

for some $x_0 \in \mathbb{R}^N$, $C > 0$, and $\varepsilon > 0$ (see Theorem 2.15 of [14]). Furthermore,

$$S_{\mu,s} = \frac{S_s}{(\mathcal{C}(N,\mu))^{\frac{1}{2^*_{\mu,s}}}}.$$

Let

$$\Psi = \frac{\nu_1}{(1+|x|^2)^{\frac{N-2s}{2}}}, \text{ where } \nu_1 = \left(\frac{S_s^{\frac{N}{2s}}\Gamma(N)}{\pi^{\frac{N}{2}}\Gamma(N)}\right).$$

Then, from [45], we have that

$$S_s\|\Psi\|^2_{2^*_s,\mathbb{R}^N} = \|\Psi\|^2_{\mathbb{R}^N} \tag{2.5}$$

and

$$\|\Psi\|^2_{\mathbb{R}^N} = \|\Psi\|^{2^*_s}_{2^*_s,\mathbb{R}^N} = S_s^{\frac{N}{2s}}.$$

Set

$$\widetilde{\Psi}(x) = S_s^{\frac{(N-\mu)(2s-N)}{4s(N-\mu+2s)}}(\mathcal{C}(N,\mu))^{\frac{2s-N}{2(N-\mu+2s)}}\Psi(x).$$

Then $\widetilde{\Psi}(x)$ is the unique minimizer for $S_{\mu,s}$ and satisfies

$$\|\widetilde{\Psi}\|^2_{\mathbb{R}^N} = \int_{\mathbb{R}^N}(I_\mu * |\widetilde{\Psi}|^{2^*_{\mu,s}})|\widetilde{\Psi}|^{2^*_{\mu,s}}\mathrm{d}x = S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}. \tag{2.6}$$

Let

$$\begin{aligned}\varphi_{\delta,z}(x) &= \delta^{\frac{2s-N}{N}}\widetilde{\Psi}\left(\frac{x-z}{\delta}\right)\\ &= \frac{a_{\mu,s}\delta^{\frac{N-2s}{2}}}{(\delta^2+|x-z|^2)^{\frac{N-2s}{2}}}\,\forall\delta>0,\ z\in\mathbb{R}^N,\end{aligned} \tag{2.7}$$

where $a_{\mu,s} = S_s^{\frac{(N-\mu)(2s-N)}{4s(N-\mu+2s)}}(\mathcal{C}(N,\mu))^{\frac{2s-N}{2(N-\mu+2s)}}\nu_1$.

Now we introduce the equation

$$(-\Delta)^s u = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u \text{ in } \mathbb{R}^N \tag{2.8}$$

and its energy functional $\mathcal{J}_\infty : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{J}_\infty(u) = \frac{1}{2}\|u\|^2_{\mathbb{R}^N} - \frac{1}{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x.$$

It follows from [28] that the positive solutions of (2.8) are unique. Furthermore, by the invariance of the scaling, the function $\varphi_{\delta,z}$ defined as (2.7) solves (2.8) and satisfies (2.6).

Let

$$\mathcal{N}_\infty := \{u \in D^{s,2}(\mathbb{R}^N)\backslash\{0\} : \langle \mathcal{J}'_\infty(u), u\rangle = 0\}.$$

Then we have that

$$\mathcal{J}_\infty(\varphi_{\delta,z}) = m_\infty := \min_{u\in\mathcal{N}_\infty} \mathcal{J}_\infty(u) = \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}.$$

Then a ground state solution of (2.8) is a nontrivial solution $u \in D^{s,2}(\mathbb{R}^N)$ satisfying

$$\mathcal{J}_\infty(u) = m_\infty \text{ and } \mathcal{J}'_\infty(u) = 0.$$

Furthermore, from [19, 20], we have the following result about a nodal solution of (2.8).

LEMMA 2.3. *If $u \in D^{s,2}(\mathbb{R}^N)$ is a nodal solution of* (2.8)*, then*

$$\mathcal{J}_\infty(u) \geq 2^{\frac{4s-\mu}{N-\mu+2s}} \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}} = 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty.$$

The following proposition indicates that there is no ground state solution to (1.1).

PROPOSITION 2.2. *Suppose that (V_1) holds. Then $m := \min_{u\in\mathcal{N}} \mathcal{J}(u) = m_\infty$, and m is not achieved.*

Proof. Step 1: To prove $m = m_\infty$. For any $u \in \mathcal{N}$, there is $t_u > 0$ satisfying $t_u u \in \mathcal{N}_\infty$. So we conclude that

$$\begin{aligned}
m_\infty \leq \mathcal{J}_\infty(t_u u) &= \frac{1}{2}\|t_u u\|^2_{\mathbb{R}^N} - \frac{1}{2\cdot 2^*_{\mu,s}} \int_{\mathbb{R}^N} (I_\mu * |t_u u|^{2^*_{\mu,s}})|t_u u|^{2^*_{\mu,s}} \mathrm{d}x \\
&= \frac{1}{2}\|t_u u\|^2 - \frac{1}{2\cdot 2^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|t_u u(x)|^{2^*_{\mu,s}} |t_u u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
&\leq \frac{1}{2}\|t_u u\|^2 + \frac{1}{2}\int_\Omega V(x)(t_u u)^2 \mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|t_u u(x)|^{2^*_{\mu,s}} |t_u u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
&= \mathcal{J}(t_u u) \leq \mathcal{J}(u),
\end{aligned}$$

which shows that $m \geq m_\infty$.

In the following, we prove that $m \leq m_\infty$. Let $\widetilde{u}_n \subset D_0^{s,2}(\Omega)$ be defined by $\widetilde{u}_n := \zeta(x)\widetilde{\varphi}_n$. Here $\widetilde{\varphi}_n(\cdot) = \varphi(\cdot - z_n)$ and $\varphi = \varphi_{1,0} \in D^{s,2}(\mathbb{R}^N)$ defined in (2.7) is a positive solution of (2.8), $\{z_n\} \subset \Omega$ with $|z_n| \to +\infty$ as $n \to +\infty$, $\zeta : \mathbb{R}^N \to [0,1]$ is defined by

$$\zeta(x) = \xi\left(\frac{|x|}{\lambda}\right), \lambda := \inf\{\tau : \mathbb{R}^N\backslash\Omega \subset \overline{B_\tau(0)}\},$$

where $B_\tau(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < \tau\}$, and $\xi(t) : \mathbb{R}^+ \cup \{0\} \to [0,1]$ is a nondecreasing function satisfying

$$\xi(t) = 0, t \leq 1 \text{ and } \xi(t) = 1, t \geq 2.$$

First, we claim that

$$\mathcal{J}(\widetilde{u}_n) \to m_\infty \text{ and } \langle \mathcal{J}'(\widetilde{u}_n), \widetilde{u}_n\rangle \to 0 \text{ as } n \to +\infty. \tag{2.9}$$

Obviously, $\|\widetilde{\varphi}_n\|_{\mathbb{R}^N} = \|\varphi\|_{\mathbb{R}^N}$ and $\widetilde{\varphi}_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to +\infty$. Hence, it follows from Lemma 2.1 that

$$\int_\Omega V(x)(\widetilde{u}_n)^2 \mathrm{d}x \to 0 \,\text{as}\, n \to +\infty.$$

Set

$$\int_{\mathbb{R}^N} A_n(x,y)\mathrm{d}y\mathrm{d}x := \int_{\mathbb{R}^N} \frac{|\widetilde{u}_n|^{2^*_{\mu,s}}|\widetilde{u}_n|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N} \frac{|\varphi(x-z_n)|^{2^*_{\mu,s}}|\varphi(y-z_n)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x,$$

where

$$A_n(x,y) = \frac{[\zeta^{2^*_{\mu,s}}(x+z_n)\zeta^{2^*_{\mu,s}}(y+z_n) - 1]|\varphi|^{2^*_{\mu,s}}|\varphi|^{2^*_{\mu,s}}}{|x-y|^\mu}.$$

Since $|z_n| \to +\infty$ as $n \to +\infty$ and

$$|x+z_n| \geq |z_n| - |x| \,\text{and}\, |y+z_n| \geq |z_n| - |y|$$

for each $x, y \in \mathbb{R}^N$, there is $N_0 \in \mathbb{N}$ satisfying

$$|x+z_n| \geq 2\lambda \,\text{and}\, |y+z_n| \geq 2\lambda, n \geq N_0.$$

Hence, $\zeta(x+z_n) = 1 = \zeta(y+z_n)$ for all $n \geq N_0$, and we have that

$$A_n(x,z) \to 0 \,\text{a.e. in}\, \mathbb{R}^N \times \mathbb{R}^N \,\text{as}\, n \to +\infty.$$

Furthermore,

$$|A_n(x,y)| \leq C\frac{|\varphi(x)|^{2^*_{\mu,s}}|\varphi(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \in L^1(\mathbb{R}^N \times \mathbb{R}^N).$$

Therefore, by the Lebesgue theorem, we conclude that

$$\int_{\mathbb{R}^N} \frac{|\widetilde{u}_n|^{2^*_{\mu,s}}|\widetilde{u}_n|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{|\varphi(x)|^{2^*_{\mu,s}}|\varphi(z)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \,\text{as}\, n \to +\infty. \tag{2.10}$$

Similar to the proof as in [1], we can show that

$$\|\widetilde{u}_n - \varphi(\cdot - z_n)\|^2_{\mathbb{R}^N} \to 0 \,\text{as}\, n \to +\infty. \tag{2.11}$$

Thus, combining with (2.10) and (2.11), the claim (2.9) holds.

For $\widetilde{u}_n \subset D_0^{s,2}(\Omega)$, arguing as in Lemma 2.2, there exists unique $t_n > 0$ such that $t_n\widetilde{u}_n \in \mathcal{N}$, and then $\langle \mathcal{J}'(t_n\widetilde{u}_n), t_n\widetilde{u}_n\rangle = 0$.

We claim that $t_n \to 1 \,\text{as}\, n \to +\infty$. Indeed, according to definition of $\widetilde{u}_n$, it is easy to see that

$$a \leq \|\widetilde{u}_n\| \leq b, a \leq \int_\Omega\int_\Omega \frac{|\widetilde{u}_n|^{2^*_{\mu,s}}|\widetilde{u}_n|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \leq b,$$

where $a, b > 0$ are constants. Hence, we can get that there is $C > 0$ such that $|t_n| \geq C$. Suppose that $t_n \to +\infty$. Thanks to $t_n\widetilde{u}_n \in \mathcal{N}$, we have that

$$\|\widetilde{u}_n\|^2 = t_n^{(2\cdot 2^*_{\mu,s}-2)}\int_\Omega\int_\Omega \frac{|\widetilde{u}_n(x)|^{2^*_{\mu,s}}|\widetilde{u}_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x + o_n(1) \,\text{as}\, n \to +\infty,$$

which is a contradiction. Thus, $\{t_n\}$ is bounded from above. So, according to the fact that $\langle \mathcal{J}'(\widetilde{u}_n), \widetilde{u}_n\rangle \to 0$ as $n\to+\infty$, it is easy to get $t_n\to 1$ as $n\to+\infty$. Then it follows from (2.9) that $\mathcal{J}(t_n\widetilde{u}_n)\to m_\infty$. Due to $t_n\widetilde{u}_n\in\mathcal{N}$, we deduce that $m\le m_\infty$.

Consequently, from the above arguments, we conclude that $m=m_\infty$.

Step 2: To prove that m is not achieved. Suppose that, by contradiction, there exists $u^\star\in\mathcal{N}$ such that $\mathcal{J}(u^\star)=m=m_\infty$ and $\mathcal{J}'(u^\star)=0$. Let $t_{u^\star}>0$ be such that $t_{u^\star}u^\star\in\mathcal{N}_\infty$. Then we have

$$
\begin{aligned}
m_\infty &\le \mathcal{J}_\infty(t_{u^\star}u^\star)=\frac{1}{2}\|t_{u^\star}u^\star\|^2_{\mathbb{R}^N}-\frac{1}{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu*|t_{u^\star}u^\star|^{2^*_{\mu,s}})|t_uu^\star|^{2^*_{\mu,s}}\mathrm{d}x\\
&=\frac{1}{2}\|t_{u^\star}u^\star\|^2-\frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|t_uu^\star(x)|^{2^*_{\mu,s}}|t_uu^\star(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\
&\le\frac{1}{2}\|t_{u^\star}u^\star\|^2+\frac{1}{2}\int_\Omega V(x)(t_{u^\star}u^\star)^2\mathrm{d}x-\frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|t_uu^\star(x)|^{2^*_{\mu,s}}|t_uu^\star(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\
&=\mathcal{J}(t_{u^\star}u^\star)\le\mathcal{J}(u)=m_\infty.
\end{aligned}
$$

Hence, we deduce that

$$
t_{u^\star}=1,\int_\Omega V(x)(u^\star)^2\mathrm{d}x=0,
$$

which implies that $u^\star$ is a minimizer of m_∞. Without loss of generality, we can assume that $u^\star\ge 0$. Therefore, by the maximum principle, we get that $u^\star>0$ in $\mathbb{R}^N$, which is impossible since $u^\star=0$ in $\mathbb{R}^N\backslash\Omega$. ∎

In the following, we cite some useful lemmas for proving our main results.

Lemma 2.4 (see [19, 42]). *Let $N>2s$ and $\mu\in(0,N)$. If $\{u_n\}$ is a bounded sequence in $L^{2^*_s}(\mathbb{R}^N)$ such that $u_n\to u$ almost everywhere in $\mathbb{R}^N$ as $n\to\infty$, then*

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(I_\mu*|u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}}\,dx-\int_{\mathbb{R}^N}(I_\mu*|u_n-u|^{2^*_{\mu,s}})|u_n-u|^{2^*_{\mu,s}}\,dx\\
&\quad\to\int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\,dx.
\end{aligned}
$$

Lemma 2.5 (see [19]). *Let $\{u_n\}$ be a bounded sequence in $D^{s,2}(\mathbb{R}^N)$ such that $u_n\to 0$ almost everywhere in $\mathbb{R}^N$ as $n\to\infty$. Denote $g(u)=(I_\mu*|u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u$. Then, for each $v\in D^{s,2}(\mathbb{R}^N)$, we have*

$$
\int_{\mathbb{R}^N}|g(u_n+v)-g(u_n)-g(v)|^{\frac{2^*_s}{2^*_s-1}}\,dx=o_n(1).
$$

By arguing as Lemma 3.4 of [20], we can obtain the following result.

Lemma 2.6. *Suppose that $\{u_n\}\subset D^{s,2}(\mathbb{R}^N)$ is a sequence of a (P.S.)c sequence for $\mathcal{J}_\infty$ such that $u_n\rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ and $u_n\nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$. Then there exist a sequence of points $\{z_n\}\subset\mathbb{R}^N$ and a sequence of positive numbers $\{\sigma_n\}$ such that*

$$
v_n(x)=\sigma_n^{\frac{N-2s}{2}}u_n(\sigma_nx+z_n) \tag{2.12}
$$

converges weakly in $D^{s,2}(\mathbb{R}^N)$ to a nontrivial solution v of (2.8).

It follows from Proposition 2.2 that (1.1) does not have any ground state solution. So we only need to find a bound state solution. For this purpose, we need to obtain the global compactness result.

Theorem 2.1. *Suppose that* (V_1) *holds. Let* $\{u_n\}\subset D_0^{s,2}(\Omega)$ *be a sequence of a (P.S.)c sequence for* $\mathcal{J}$, *that is,*

$$\mathcal{J}(u_n)\to c \text{ and } \mathcal{J}'(u_n)\to 0 \text{ as } n\to+\infty. \tag{2.13}$$

Then there exist a number $l\in\mathbb{N}:=\{0,1,2,\dots\}$; l *sequences of numbers* $\{\sigma_n^i\}\subset\mathbb{R}^+$; *points* $\{z_n^i\}\subset\mathbb{R}^N$, $1\le i\le l$, $l+1$; *and sequences of functions* $\{u_n^{(j)}\}\subset D^{s,2}(\mathbb{R}^N), 0\le j\le l$, *such that for some subsequence, still denoted by* $\{u_n\}$,

$$\begin{aligned}
&\text{(a) } u_n(x)=u_n^{(0)}(x)+\sum_{i=1}^{l}(\sigma_n^i)^{-\frac{N-2s}{2}}u_n^{(i)}\left(\frac{\cdot-z_n^i}{\sigma_n^i}\right),\\
&\text{(b) } u_n^{(0)}\rightharpoonup u^{(0)} \text{ in } D_0^{s,2}(\Omega) \text{ as } n\to+\infty,\\
&\text{(c) } u_n^{(i)}\rightharpoonup u^{(i)}\ne 0 \text{ in } D^{s,2}(\mathbb{R}^N) \text{ as } n\to+\infty, 1\le i\le l,
\end{aligned}\tag{2.14}$$

where $u^{(0)},u^{(i)}(1\le i\le l)$ *satisfy*

(2.15)

$$(-\Delta)^s u^{(0)}+V(x)u^{(0)}=\Big(\int_\Omega\frac{|u^{(0)}(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\Big)|u^{(0)}|^{2^*_{\mu,s}-2}u^{(0)}, x\in\Omega; u=0, x\in\mathbb{R}^N\backslash\Omega,$$

$$(-\Delta)^s u^{(i)}=(I_\mu*|u^{(i)}|^{2^*_{\mu,s}})|u^{(i)}|^{2^*_{\mu,s}-2}u^{(i)}, x\in\mathbb{R}^N, 1\le i\le l. \tag{2.16}$$

Moreover, we have

$$\|u_n-u^{(0)}-\sum_{i=1}^{l}(\sigma_n^i)^{-\frac{N-2s}{2}}u^{(i)}\left(\frac{\cdot-z_n^i}{\sigma_n^i}\right)\|^2\to 0 \text{ as } n\to+\infty \tag{2.17}$$

and

$$\mathcal{J}(u_n)\to\mathcal{J}(u^{(0)})+\sum_{i=1}^{l}\mathcal{J}_\infty(u^{(i)}) \text{ as } n\to+\infty. \tag{2.18}$$

Proof. First, we prove that $\{u_n\}$ is bounded in $D_0^{s,2}(\Omega)$. It follows from (2.13) that

$$\begin{aligned}
c+o_n(1)+o_n(1)\|u_n\|&=\mathcal{J}(u_n)-\frac{1}{2\cdot 2^*_{\mu,s}}\langle\mathcal{J}'(u_n),u_n\rangle\\
&\le\left(\frac12-\frac{1}{2\cdot 2^*_{\mu,s}}\right)\|u_n\|^2+\left(\frac12-\frac{1}{2\cdot 2^*_{\mu,s}}\right)\int_\Omega V(x)u_n^2\mathrm{d}x,
\end{aligned}$$

which, combined with $V(x)\ge 0$, shows that $\{u_n\}$ is bounded in $D_0^{s,2}(\Omega)$. So there is $u^{(0)}\in D_0^{s,2}(\Omega)$ such that, up to a subsequence, still denoted by $\{u_n\}$,

$$u_n\rightharpoonup u^{(0)} \text{ in } D_0^{s,2}(\Omega),\ u_n\to u^{(0)} \text{ a.e. in } \Omega. \tag{2.19}$$

For any $\psi\in C_0^\infty(\mathbb{R}^N)$, by Lemmas 2.1 and 2.5, we have that

$$\begin{aligned}
\langle\mathcal{J}'(u_n),\psi\rangle&=(u_n,\psi)+\int_\Omega V(x)u_n\psi\mathrm{d}x-\int_\Omega\int_\Omega\frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}-2}u_n(y)\psi(y)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\
&=(u^{(0)},\psi)+\int_\Omega V(x)u^{(0)}\psi\mathrm{d}x\\
&\quad-\int_\Omega\int_\Omega\frac{|u^{(0)}(x)|^{2^*_{\mu,s}}|u^{(0)}(y)|^{2^*_{\mu,s}-2}u^{(0)}(y)\psi(y)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x+o_n(1)\\
&=\langle\mathcal{J}'(u^{(0)}),\psi\rangle+o_n(1),
\end{aligned}$$

which shows that $\langle \mathcal{J}'(u^{(0)}), \psi\rangle = 0$. That is, $u^{(0)}$ satisfies (2.15).

Let

$$v_n^{(1)}(x) = \begin{cases} (u_n - u^{(0)})(x), x \in \Omega, \\ 0, x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{2.20}$$

Then it follows from (2.19) that $v_n^{(1)} \rightharpoonup 0$ in $D_0^{s,2}(\Omega)$ as $n \to +\infty$. Thanks to Lemma 2.1, we have

$$\int_{\mathbb{R}^N} V(x)(v_n^{(1)})^2 \mathrm{d}x \to 0 \text{ as } n \to +\infty. \tag{2.21}$$

Furthermore, by Lemmas 2.1 and 2.4, we have that

(2.22)

$$\begin{aligned}
&\|v_n^{(1)}\|_{\mathbb{R}^N}^2 = \|v_n^{(1)}\|^2 = \|u_n\|^2 - \|u^{(0)}\|^2 + o_n(1), \\
&\mathcal{J}_\infty(v_n^{(1)}) = \frac{1}{2}\|v_n^{(1)}\|_{\mathbb{R}^N}^2 + \frac{1}{2}\int_\Omega V(x)(v_n^{(1)})^2 \mathrm{d}x \\
&\qquad - \frac{1}{2 \cdot 2^*_{\mu,s}} \int_{\mathbb{R}^N} (I_\mu * |v_n^{(1)}|^{2^*_{\mu,s}})|v_n^{(1)}|^{2^*_{\mu,s}} \mathrm{d}x + o_n(1) \\
&\qquad = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_\Omega V(x)u_n^2 \mathrm{d}x - \frac{1}{2 \cdot 2^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
&\qquad - \frac{1}{2}\|u^{(0)}\|^2 - \frac{1}{2}\int_\Omega V(x)(u^{(0)})^2 \mathrm{d}x \\
&\qquad + \frac{1}{2 \cdot 2^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|u^{(0)}(x)|^{2^*_{\mu,s}}|u^{(0)}(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x + o_n(1) \\
&\qquad = \mathcal{J}(u_n) - \mathcal{J}(u^{(0)}) + o_n(1).
\end{aligned}$$

If $v_n^{(1)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$, then the theorem is proved with $l = 0$.

If $v_n^{(1)} \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$ then, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, thanks to Lemma 2.1 and (2.13), we have that

$$\begin{aligned}
\langle \mathcal{J}'_\infty(v_n^{(1)}), \psi\rangle &= (v_n^{(1)}, \psi)_{\mathbb{R}^N} + \int_\Omega V(x)v_n^{(1)}\psi \mathrm{d}x + o_n(1)\|\psi\|_{\mathbb{R}^N} \\
&\quad - \int_{\mathbb{R}^N} (I_\mu * |v_n^{(1)}|^{2^*_{\mu,s}})|v_n^{(1)}|^{2^*_{\mu,s}-2}v_n^{(1)}\psi \mathrm{d}x \\
&= \langle \mathcal{J}'(v_n^{(1)}), \psi\rangle + o_n(1)\|\psi\|_{\mathbb{R}^N} \\
&= \langle \mathcal{J}'(u_n), \psi\rangle - \langle \mathcal{J}'(u^{(0)}), \psi\rangle + o_n(1) \\
&= o_n(1),
\end{aligned} \tag{2.23}$$

which shows that $\mathcal{J}'_\infty(v_n^{(1)}) \to 0$ as $n \to +\infty$. Hence, $\{v_n^{(1)}\}$ is a Palais–Smale sequence for $\mathcal{J}_\infty$ and satisfies

$$v_n^{(1)} \rightharpoonup 0 \text{ in } D^{s,2}(\mathbb{R}^N),\ v_n^{(1)} \nrightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Then, by Lemma 2.6, there exist $\{z_n^1\} \subset \mathbb{R}^N$, $\{\sigma_n^1\} \subset \mathbb{R}^+$, and $u^{(1)} \in D^{s,2}(\mathbb{R}^N)$ such that

$$\begin{aligned}
&u_n^{(1)} := (\sigma_n^1)^{\frac{N-2s}{2}} v_n^{(1)}(\sigma_n^1 \cdot + z_n^1), \\
&u_n^{(1)} \rightharpoonup u^{(1)} \text{ in } D^{s,2}(\mathbb{R}^N), \\
&\mathcal{J}'_\infty(u^{(1)}) = 0,\ u^{(1)} \neq 0.
\end{aligned}$$

Thus, $u^{(1)}$ is a nontrivial solution of (2.16). Moreover, we have

$$\|v_n^{(1)}\|_{\mathbb{R}^N}^2 = \|u_n^{(1)}\|_{\mathbb{R}^N}^2 = \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u_n^{(1)} - u^{(1)}\|_{\mathbb{R}^N}^2 + o_n(1),$$
$$\mathcal{J}_\infty(v_n^{(1)}) = \mathcal{J}_\infty(u_n^{(1)}) = \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u_n^{(1)} - u^{(1)}) + o_n(1).$$

Combining with above equality and (2.22), we conclude that

$$\begin{aligned}\|u_n\|^2 &= \|v_n^{(1)}\|_{\mathbb{R}^N}^2 + \|u^{(0)}\|^2 + o_n(1)\\ &= \|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u_n^{(1)} - u^{(1)}\|_{\mathbb{R}^N}^2 + o_n(1),\\ \mathcal{J}(u_n) &= \mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(v_n^{(1)}) + o_n(1)\\ &= \mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u_n^{(1)} - u^{(1)}) + o_n(1).\end{aligned} \tag{2.24}$$

Let $v_n^{(2)} := u_n^{(1)} - u^{(1)}$. If $v_n^{(2)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$, then the theorem is proved with $l = 1$.

If $v_n^{(2)} \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$, then, similarly, we can conclude that $\{v_n^{(2)}\}$ is a Palais–Smale sequence for $\mathcal{J}_\infty$ and satisfies

$$v_n^{(2)} \rightharpoonup 0 \text{ in } D^{s,2}(\mathbb{R}^N),\ v_n^{(2)} \nrightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Then it follows from Lemma 2.6 that there are $\{z_n^2\} \subset \mathbb{R}^N$, $\{\sigma_n^2\} \subset \mathbb{R}^+$, and $u^{(2)} \in D^{s,2}(\mathbb{R}^N)$ so that

$$\begin{aligned}&u_n^{(2)} := (\sigma_n^2)^{\frac{N-2s}{2}} v_n^{(2)}(\sigma_n^2 \cdot + z_n^2),\\ &u_n^{(2)} \rightharpoonup u^{(2)} \text{ in } D^{s,2}(\mathbb{R}^N),\\ &\mathcal{J}_\infty'(u^{(2)}) = 0,\ u^{(2)} \neq 0,\end{aligned}$$

which implies that $u^{(2)}$ is a nontrivial solution of (2.16). Moreover, one has

$$\|v_n^{(2)}\|_{\mathbb{R}^N}^2 = \|u_n^{(2)}\|_{\mathbb{R}^N}^2 = \|u^{(2)}\|_{\mathbb{R}^N}^2 + \|u_n^{(2)} - u^{(2)}\|_{\mathbb{R}^N}^2 + o_n(1),$$
$$\mathcal{J}_\infty(v_n^{(2)}) = \mathcal{J}_\infty(u_n^{(2)}) = \mathcal{J}_\infty(u^{(2)}) + \mathcal{J}_\infty(u_n^{(2)} - u^{(2)}) + o_n(1).$$

Together with (2.24), we conclude that

$$\begin{aligned}\|u_n\|^2 &= \|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u_n^{(1)} - u^{(1)}\|_{\mathbb{R}^N}^2 + o_n(1)\\ &= \|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|v_n^{(2)}\|_{\mathbb{R}^N}^2 + o_n(1)\\ &= \|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u^{(2)}\|_{\mathbb{R}^N}^2 + \|v_n^{(2)} - u^{(2)}\|_{\mathbb{R}^N}^2 + o_n(1),\\ \mathcal{J}(u_n) &= \mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u_n^{(1)} - u^{(1)}) + o_n(1)\\ &= \mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(v_n^{(2)}) + o_n(1)\\ &= \mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u^{(2)}) + \mathcal{J}_\infty(v_n^{(2)} - u^{(2)}) + o_n(1).\end{aligned} \tag{2.25}$$

Iterating the above procedures, we can obtain sequences $\{u_n^{(k-1)}\}$ in this way. Let $v_k^{(k)} := u_n^{(k-1)} - u^{(k-1)}$. If $v_n^{(k)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$, then the theorem is proved with $l = k$.

If $v_n^{(k)} \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$, then, arguing as before, $\{v_n^{(k)}\}$ is a Palais–Smale sequence for $\mathcal{J}_\infty$ such that

$$v_n^{(k)} \rightharpoonup 0 \text{ in } D^{s,2}(\mathbb{R}^N),\ v_n^{(k)} \nrightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Then, according to Lemma 2.6, there exist $\{z_n^k\} \subset \mathbb{R}^N$, $\{\sigma_n^k\} \subset \mathbb{R}^+$, and $u^{(k)} \in D^{s,2}(\mathbb{R}^N)$ satisfying

$$\begin{aligned}
&u_n^{(k)} := (\sigma_n^k)^{\frac{N-2s}{2}} v_n^{(k)}(\sigma_n^k \cdot + z_n^k),\\
&u_n^{(k)} \rightharpoonup u^{(k)} \text{ in } D^{s,2}(\mathbb{R}^N),\\
&\mathcal{J}'_\infty(u^{(k)}) = 0,\ u^{(k)} \neq 0.
\end{aligned}$$

Thus, $u^{(k)}$ is a nontrivial solution of (2.16). Furthermore,

$$\begin{aligned}
\|u_n\| =&\|u^{(0)}\|^2 + \Sigma_{i=1}^k \|u^{(i)}\|_{\mathbb{R}^N}^2 + \|u_n^{(k)} - u^{(k)}\|_{\mathbb{R}^N}^2 + o_n(1),\\
\mathcal{J}(u_n) =&\mathcal{J}(u^{(0)}) + \Sigma_{i=1}^k \mathcal{J}_\infty(u^{(i)}) + \mathcal{J}_\infty(u_n^{(k)} - u^{(k)}) + o_n(1).
\end{aligned} \tag{2.26}$$

Thanks to

$$0 = \langle \mathcal{J}'_\infty(u^{(i)}), u^{(i)} \rangle = \|u^{(i)}\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N} (I_\mu * |u^{(i)}|^{2^*_{\mu,s}})|u^{(i)}|^{2^*_{\mu,s}} \mathrm{d}x, i \in \{1, 2 \ldots, k\}$$

and the definition of $S_{\mu,s}$, we obtain that $\|u^{(i)}\|_{\mathbb{R}^N} \geq S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}, i = 1, 2, \ldots, k$. Then we conclude that the iteration must terminate at a finite index $l \geq 1$, that is, $v_n^{(l+1)} := u_n^{(l)} - u^{(l)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$. Then we have

$$\begin{aligned}
\|u_n\| &= \|u^{(0)}\|^2 + \Sigma_{i=1}^l \|u^{(i)}\|_{\mathbb{R}^N}^2 + o_n(1),\\
\mathcal{J}(u_n) &= \mathcal{J}(u^{(0)}) + \Sigma_{i=1}^l \mathcal{J}_\infty(u^{(i)}) + o_n(1).
\end{aligned}$$

Moreover, it is easy to obtain from the above discussion that

$$\begin{aligned}
u_n = &(u^{(0)} + o_n(1)) + (\sigma_n^1)^{-\frac{N-2s}{2}}(u^{(1)} + o_n(1))\Big(\frac{x - z_n^1}{\sigma_n^1}\Big)\\
&+ (\sigma_n^1\sigma_n^2)^{-\frac{N-2s}{2}}(u^{(2)} + o_n(1))\Big(\frac{x - z_n^1 - \sigma_n^1 z_n^2}{\sigma_n^1\sigma_n^2}\Big)\\
&+ (\sigma_n^1\sigma_n^2\sigma_n^3)^{-\frac{N-2s}{2}}(u^{(3)} + o_n(1))\Big(\frac{x - z_n^1 - \sigma_n^1 z_n^2 - \sigma_n^1\sigma_n^2 z_n^3}{\sigma_n^1\sigma_n^2\sigma_n^3}\Big)\\
&+ \cdots\\
&+ (\sigma_n^1\sigma_n^2\sigma_n^3\cdots\sigma_n^l)^{-\frac{N-2s}{2}}(u^{(l)} + o_n(1))\\
&\quad\left(\frac{x - z_n^1 - \sigma_n^1 z_n^2 - \sigma_n^1\sigma_n^2 z_n^3 - \cdots - \sigma_n^1\sigma_n^2\sigma_n^3\cdots\sigma_n^{l-1} z_n^l}{\sigma_n^1\sigma_n^2\sigma_n^3\cdots\sigma_n^l}\right).
\end{aligned}$$

So it follows from rewriting the notations that (2.14)–(2.16) are satisfied. ∎

COROLLARY 2.1. *Suppose that* (V_1) *holds. Let* $\{u_n\} \subset D_0^{s,2}(\Omega)$ *be a nonnegative sequence such that*

$$\mathcal{J}(u_n) \to m,\ \langle \mathcal{J}'(u_n), u_n \rangle = 0 \ as\, n \to +\infty. \tag{2.27}$$

Then we have

$$u_n = w_n + \varphi_{\delta_n, z_n}, \tag{2.28}$$

where $\{w_n\} \subset D^{s,2}(\mathbb{R}^N)$ *such that* $w_n \to 0$ *in* $D^{s,2}(\mathbb{R}^N)$ *and* φ_{δ_n,z_n} *defined in* (2.7) *is the positive function realizing* m_∞.

Proof. It follows from (2.27) that $\{u_n\}$ is a minimizing sequence for $\mathcal{J}|_{\mathcal{N}}$. Then it follows from the variational principle (see [53]) that there is a sequence $\{v_n\}\subset\mathcal{N}$ satisfying

$$\mathcal{J}(v_n)\to m,\ \mathcal{J}'(v_n)-\varsigma_n\mathcal{G}'(v_n)\to 0, \|u_n-v_n\|\to 0 \text{ as } n\to+\infty, \tag{2.29}$$

where $\varsigma_n\in\mathbb{R}$. We may assume that $v_n\geq 0$.

Next, we prove that $\mathcal{J}'(v_n)\to 0$ as $n\to+\infty$. From (2.29), one has that

$$\langle \mathcal{J}'(v_n),v_n\rangle-\varsigma_n\langle\mathcal{G}'(v_n),v_n\rangle=o_n(1)\|v_n\|.$$

It follows from (2.4), (2.5), and the fact that $\{v_n\}\subset\mathcal{N}$ that $\langle\mathcal{G}'(v_n),v_n\rangle<\widetilde{C}<0$. Therefore, thanks to $\{v_n\}\subset\mathcal{N}$ and $\mathcal{J}(v_n)\to m$, it is easy to see that $\{v_n\}$ is bounded. So $\varsigma_n\to 0$ as $n\to+\infty$. For any $\phi\in D_0^{s,2}(\Omega)$,

$$\begin{aligned}\langle\mathcal{G}'(v_n),\phi\rangle&=2(v_n,\phi)+2\int_\Omega V(x)v_n\phi\mathrm{d}x\\&\quad-2\cdot 2^*_{\mu,s}\int_\Omega\int_\Omega\frac{|v_n(x)|^{2^*_{\mu,s}}|v_n(y)|^{2^*_{\mu,s}-2}v_n(y)\phi(y)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Thus, according to the boundedness of $|V|_{\frac{N}{2s}}$, using the Hölder inequality, we deduce that

$$|\langle\mathcal{G}'(v_n),\phi\rangle|\leq\left(C_1\|v_n\|+C_2\|v_n\|^{\frac{3N+2s-2\mu}{N-2s}}\right)\|\phi\|,$$

which, combined with the boundedness of $\{v_n\}$, shows that $\mathcal{G}'(v_n)$ is bounded. Then it is easy to obtain that $\mathcal{J}'(v_n)\to 0$ as $n\to+\infty$.

Now, for any $\phi\in D_0^{s,2}(\Omega)$, it is easy to get that

$$\langle\mathcal{J}'(u_n)-\mathcal{J}'(v_n),\phi\rangle\to 0 \text{ as } n\to+\infty,$$

from which we conclude that $\mathcal{J}'(u_n)\to 0$ as $n\to+\infty$.

By Theorem 2.1, there exist a number $l\in\mathbb{N}$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that (2.15)–(2.18) hold.

If $u^{(0)}\not\equiv 0$ and $l\geq 1$, then, thanks to (2.15)–(2.16), we have

$$\langle\mathcal{J}'(u^{(0)}),u^{(0)}\rangle=0, \langle\mathcal{J}'_\infty(u^{(i)}),u^{(i)}\rangle=0, 1\leq i\leq l,$$

which shows that $u^{(0)}\in\mathcal{N}, u^{(i)}\in\mathcal{N}_\infty, 1\leq i\leq l$. Hence, we have that $\mathcal{J}(u^{(0)})\geq m=m_\infty, \mathcal{J}_\infty(u^{(i)})\geq m_\infty, 1\leq i\leq l$. We deduce that

$$m=\mathcal{J}(u^{(0)})+\sum_{i=1}^{l}\mathcal{J}_\infty(u^{(i)})\geq(l+1)m_\infty\geq 2m_\infty,$$

which contradicts with the fact that $m=m_\infty$.

If $u^{(0)}\equiv 0$ and $l\geq 2$, then, similar to the above discussion, we obtain a contradiction:

$$m=\sum_{i=1}^{l}\mathcal{J}_\infty(u^{(i)})\geq lm_\infty\geq 2m_\infty.$$

If $u^{(0)}\not\equiv 0$ and $l=0$, then $u^{(0)}$ is a ground state solution of (1.1), which contradicts with Proposition 2.2.

If $u^{(0)} \equiv 0$ and $l = 0$, then we also get a contradiction due to the fact that $\mathcal{J}(u^{(0)}) = m$.

According to above arguments, we must have that $u^{(0)} \equiv 0$ and $l = 1$. That is, $u^{(1)}$ satisfies (2.8), and then $\mathcal{J}_\infty(u^{(1)}) = m = m_\infty$. This fact shows that $u^{(1)}$ is a ground state solution of (2.8). Then it follows from Lemma 2.3 and the nonnegativity of $\{u_n\}$ that $u^{(1)} \geq 0$. Arguing as in Corollary 4.2 of [18], we can get $u^{(1)} > 0$. Consequently, it follows from the above discussion and the uniqueness of the positive solution of (2.8) that (2.28) holds. □

Corollary 2.2. *Suppose that* (V_1) *holds. Let* $\{u_n\} \subset D_0^{s,2}(\Omega)$ *be a nonnegative sequence of a (P.S.)c sequence for* $\mathcal{J}$*, that is,*

$$\mathcal{J}(u_n) \to c \text{ and } \mathcal{J}'(u_n) \to 0 \text{ as } n \to +\infty. \tag{2.30}$$

If $c \in (m_\infty, 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty)$*, then the functional* $\mathcal{J}$ *satisfies the (P.S.)c condition.*

Proof. According to Theorem 2.1, there are $l \in \mathbb{N}$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that (2.15)–(2.18) hold. Thanks to $0 < \frac{4s-\mu}{N-\mu+2s} < 1$ and $c \in (m_\infty, 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty)$, similar to the arguments of Corollary 2.1, we can conclude that $u^{(0)} \not\equiv 0$ and $l = 0$. So it follows from (2.17) that $u_n \to u^{(0)}$ in $D_0^{s,2}(\Omega)$. □

3. Main technique and some basic estimates. Let $\varphi_{\delta,z}$ defined by (2.7) be the ground state solution of (2.8). Without any loss of generality, we assume that $0 \in \mathbb{R}^N \backslash \Omega$. Then $\lambda = \inf\{\tau : \mathbb{R}^N \backslash \Omega \subset \overline{B_\tau(0)}\} > 0$. Let

$$v_\lambda := \zeta(x)\varphi_{\delta,z} = \xi\Big(\frac{|x|}{\lambda}\Big)\varphi_{\delta,z},$$

where ζ, ξ are defined as in Proposition 2.2. Define $\mathcal{K}_\lambda : \mathbb{R}^N \times \mathbb{R}^+ \to D^{s,2}(\mathbb{R}^N)$ by

$$\mathcal{K}_\lambda(z,\delta) = t_{v_\lambda} v_\lambda,$$

where $t_{v_\lambda} > 0$ satisfies $\langle \mathcal{J}'(t_{v_\lambda} v_\lambda), t_{v_\lambda} v_\lambda \rangle = 0$. According to the definitions, v_λ and $\mathcal{K}_\lambda(z,\delta)$ can be seen as elements in $D_0^{s,2}(\Omega)$ and $L^{2_s^*}(\Omega)$. Moreover, we have that

$$\|\mathcal{K}_\lambda(z,\delta)\| = \|\mathcal{K}_\lambda(z,\delta)\|_{\mathbb{R}^N},\ \|v_\lambda\| = \|v_\lambda\|_{\mathbb{R}^N},$$
$$|\mathcal{K}_\lambda(z,\delta)|_{2_s^*} = |\mathcal{K}_\lambda(z,\delta)|_{2^*,\mathbb{R}^N},\ |v_\lambda|_{2_s^*} = |v_\lambda|_{2_s^*,\mathbb{R}^N}.$$

Lemma 3.1. *Suppose that* (V_1) *holds with* $|V|_{\frac{N}{2s}} \neq 0$*. Then* $\mathcal{K}_\lambda(z,\delta)$ *satisfies that*
(a) $\mathcal{K}_\lambda(z,\delta)$ *is continuous in* (z,δ) *for every* λ*;*
(b) $\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \to m_\infty$ *and* $\langle \mathcal{J}'(\mathcal{K}_\lambda(z,\delta)), \mathcal{K}_\lambda(z,\delta)\rangle \to 0$ *as* $|z| \to +\infty$*, uniformly for every bounded* λ*, and bounded* δ *away from* 0*;*
(c) *as* $\lambda \to 0$*,* $\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \to m_\infty$ *and* $\langle \mathcal{J}'(\mathcal{K}_\lambda(z,\delta)), \mathcal{K}_\lambda(z,\delta)\rangle \to 0$ *as* $\delta \to 0$ *or* $\delta \to +\infty$*, uniformly in* $z \in \mathbb{R}^N$*.*

Proof. (a) obviously holds. By similar arguments as in the proof of Proposition 2.2, we easily obtain (b). In the following, we prove (c):

$$
\begin{aligned}
&\|v_\lambda-\varphi_{\delta,z}\|^2_{\mathbb{R}^N}\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\left(\xi(\frac{|x|}{\lambda})-\xi(\frac{|y|}{\lambda})\right)\varphi_{\delta,0}(x-z)+\left(\xi(\frac{|y|}{\lambda})-1\right)\left(\varphi_{\delta,0}(x-z)-\varphi_{\delta,0}(y-z)\right)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\le 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\xi(\frac{|x|}{\lambda})-\xi(\frac{|y|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x-z)\right|^2}{|x-y|^{N+2s}}dxdy\\
&\quad+2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\xi(\frac{|y|}{\lambda})-1\right|^2\left|\varphi_{\delta,0}(x-z)-\varphi_{\delta,0}(y-z)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&:=2(I_1+I_2),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\xi(\frac{|x|}{\lambda})-\xi(\frac{|y|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x-z)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x,\\
I_2&:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\xi(\frac{|y|}{\lambda})-1\right|^2\left|\varphi_{\delta,0}(x-z)-\varphi_{\delta,0}(y-z)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

First, we claim $I_2\to 0$ as $\lambda\to 0$. In fact,

$$
I_2=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\xi(\frac{|y+z|}{\lambda})-1\right|^2\left|\varphi_{\delta,0}(x)-\varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.
$$

By the definition of ξ, we have that

$$
\frac{\left|\xi(\frac{|y+z|}{\lambda})-1\right|^2\left|\varphi_{\delta,0}(x)-\varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}}\le 4\frac{\left|\varphi_{\delta,0}(x)-\varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}}\in L^1(\mathbb{R}^N\times\mathbb{R}^N),
$$

$$
\frac{\left|\xi(\frac{|y+z|}{\lambda})-1\right|^2\left|\varphi_{\delta,0}(x)-\varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}}\to 0\text{ a.e. in }\mathbb{R}^N\times\mathbb{R}^N\text{ as }\lambda\to 0.
$$

Hence, the Lebesgue theorem ensures that

$$
I_2\to 0\text{ as }\lambda\to 0\text{ for every }z\in\mathbb{R}^N. \tag{3.1}
$$

Next, arguing as in Lemma 4.1 of [1] (see also Lemma 2.3 of [54]), we prove that $I_1\to 0$ as $\lambda\to 0$ for $z\in\mathbb{R}^N$. Let $\mathbb{R}^N\times\mathbb{R}^N=\Pi_1\cup\Pi_2\cup\Pi_3$. It is easy to see that

$$
\begin{aligned}
I_1&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&=\sum_{i=1}^{3}\int_{\Pi_i}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x,
\end{aligned}
$$

where

$$
\begin{aligned}
&\Pi_1:=(\mathbb{R}^N\backslash B_{2\lambda}(-z))\times(\mathbb{R}^N\backslash B_{2\lambda}(-z)),\ \Pi_2:=B_{2\lambda}(-z)\times\mathbb{R}^N,\\
&\Pi_3:=(\mathbb{R}^N\backslash B_{2\lambda}(-z))\times B_{2\lambda}(-z).
\end{aligned}
$$

For any $(x,y)\in\Pi_1$, $|x+z|\geq 2\lambda, |y+z|\geq 2\lambda$; hence, $\xi(\frac{|x+z|}{\lambda})=\xi(\frac{|y+z|}{\lambda})=1$. Then we have that

$$\int_{\Pi_1}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x=0. \tag{3.2}$$

Denote

$$\begin{aligned}&\int_{\Pi_2}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&=\int_{B_{2\lambda}(-z)}\int_{B_\lambda(x)}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&\quad+\int_{B_{2\lambda}(-z)}\int_{\mathbb{R}^N\setminus B_\lambda(x)}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

On the one hand, using the mean value theorem, we conclude that

$$\begin{aligned}&\int_{B_{2\lambda}(-z)}\int_{B_\lambda(x)}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&\leq\frac{C_1}{(2-2s)\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx.\end{aligned}\tag{3.3}$$

On the other hand, we obtain that

$$\begin{aligned}&\int_{B_{2\lambda}(-z)}\int_{\mathbb{R}^N\setminus B_\lambda(x)}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&=4\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2\int_{\mathbb{R}^N\setminus B_\lambda(x)}\frac{1}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&\leq\frac{C_2}{s\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx.\end{aligned}\tag{3.4}$$

Combining with (3.3) and (3.4), we have that

$$\int_{\Pi_2}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\leq\frac{C_3}{\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx. \tag{3.5}$$

Define

$$\mathcal{A}:=\{y\in B_{2\lambda}(-z):|x-y|\leq\lambda\},\mathcal{A}^c:=\{y\in B_{2\lambda}(-z):|x-y|>\lambda\}.$$

Then

$$\begin{aligned}&\int_{\Pi_3}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&=\int_{\mathbb{R}^N\setminus B_{2\lambda}(-z)}\int_{\mathcal{A}}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N\setminus B_{2\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Arguing as before, we can obtain that

$$
\begin{aligned}
&= \int_{\mathbb{R}^N \setminus B_{2\lambda}(-z)} \int_{\mathcal{A}} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\le \frac{C_4}{\lambda^{2s}} \int_{B_{3\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx.
\end{aligned} \tag{3.6}
$$

Choose $\gamma > 4$. It is easy to see that

$$
(\mathbb{R}^N \setminus B_{2\lambda}(-z)) \times B_{2\lambda}(-z) \subset [(B_{\gamma\lambda}(-z) \times B_{2\lambda}(-z)) \cup (\mathbb{R}^N \setminus B_{\gamma\lambda}(-z) \times B_{2\lambda}(-z))].
$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{2\lambda}(-z)} \int_{\mathcal{A}^c} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\le \int_{B_{\gamma\lambda}(-z)} \int_{\mathcal{A}^c} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N \setminus B_{\gamma\lambda}(-z)} \int_{\mathcal{A}^c} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{3.7}
$$

By direct computation, one gets

$$
\begin{aligned}
&\int_{B_{\gamma\lambda}(-z)} \int_{\mathcal{A}^c} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\le \frac{C_5}{s\lambda^{2s}} \int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx.
\end{aligned} \tag{3.8}
$$

If $(x,y) \in \mathbb{R}^N \setminus B_{\gamma\lambda}(-z)) \times B_{2\lambda}(-z)$, then we have

$$
|x-y| \ge |x+z| - |z+y| \ge \frac{|x+z|}{2} + \frac{\gamma\lambda}{2} - 2\lambda \ge \frac{|x+z|}{2}.
$$

So we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{\gamma\lambda}(-z)} \int_{\mathcal{A}^c} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\le \frac{C_6}{\gamma^N} \left(\int_{\mathbb{R}^N \setminus B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}.
\end{aligned} \tag{3.9}
$$

Hence, it follows from (3.7)–(3.9) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{2\lambda}(-z)} \int_{\mathcal{A}^c} \frac{\left|\xi\left(\frac{|x+z|}{\lambda}\right) - \xi\left(\frac{|y+z|}{\lambda}\right)\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\le \frac{C_6}{\gamma^N} \left(\int_{\mathbb{R}^N \setminus B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} + \frac{C_7}{\lambda^{2s}} \int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx.
\end{aligned} \tag{3.10}
$$

Therefore, combining with (3.2), (3.5), (3.6), and (3.10), we conclude that

$$\begin{aligned} I_1 \leq & \frac{C_3}{\lambda^{2s}} \int_{B_{2\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx + \frac{C_4}{\lambda^{2s}} \int_{B_{3\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx \\ & + \frac{C_7}{\lambda^{2s}} \int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx + \frac{C_6}{\gamma^N} \left(\int_{\mathbb{R}^N \setminus B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\ \leq & \frac{C_8}{\lambda^{2s}} \int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^2 dx + \frac{C_6}{\gamma^N} \left(\int_{\mathbb{R}^N \setminus B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\ \leq & C_9 \gamma^{2s} \left(\int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} + \frac{C_{10}}{\gamma^N}. \end{aligned}$$

Given $\varepsilon > 0$, we can fix γ large enough such that $\frac{C_{10}}{\gamma^N} < \frac{\varepsilon}{2}$. So

$$I_1 \leq C_9 \gamma^{2s} \left(\int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} + \frac{\varepsilon}{2}.$$

Now let us fix λ small enough such that

$$\int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*} dx < \left(\frac{\varepsilon}{2}\right)^{\frac{2}{2_s^*}} \frac{1}{C_9 \gamma^{2s}}.$$

From the above arguments, we get that $I_1 \leq \varepsilon$ uniformly in z for λ small enough, which shows that

$$I_1 \to 0 \text{ as } \lambda \to 0 \text{ for } z \in \mathbb{R}^N.$$

Then, together with (3.2), we deduce that

$$\|v_\lambda\|^2 = \|v_\lambda\|^2_{\mathbb{R}^N} \to \|\varphi_{\delta,z}\|^2_{\mathbb{R}^N} \text{ as } \lambda \to 0 \ \forall (z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{3.11}$$

It follows from simple calculation that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\lambda(x)|^{2^*_{\mu,s}} |v_\lambda(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi_{\delta,z}(x)|^{2^*_{\mu,s}} |\varphi_{\delta,z}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \mathrm{d}y\mathrm{d}x \\ = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\lambda(x)|^{2^*_{\mu,s}} |v_\lambda(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi_{\delta,0}(x-z)|^{2^*_{\mu,s}} |\varphi_{\delta,0}(y-z)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \mathrm{d}y\mathrm{d}x \\ = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left[\left| \xi\left(\frac{|x+z|}{\lambda}\right) \right|^{2^*_{\mu,s}} \left| \xi\left(\frac{|y+z|}{\lambda}\right) \right|^{2^*_{\mu,s}} - 1 \right] |\varphi_{\delta,0}(x)|^{2^*_{\mu,s}} |\varphi_{\delta,0}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \mathrm{d}y\mathrm{d}x \\ := & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Pi_\lambda(x,y) \mathrm{d}y\mathrm{d}x. \end{aligned}$$

Since

$$\Pi_\lambda(x,y) \leq C \frac{|\varphi_{\delta,0}(x)|^{2^*_{\mu,s}} |\varphi_{\delta,0}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$$

and

$$\Pi_\lambda(x,y) \to 0 \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \text{ as } \lambda \to 0,$$

using the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\lambda(x)|^{2^*_{\mu,s}}|v_\lambda(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_{\delta,z}(x)|^{2^*_{\mu,s}}|\varphi_{\delta,z}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x \tag{3.12}$$

uniformly in $z\in\mathbb{R}^N$ as $\lambda\to 0$.

Thanks to

$$\begin{aligned}|v_\lambda-\varphi_{\delta,z}|^{2^*_s}_{2^*_s,\mathbb{R}^N} &= \int_{B_{2\lambda}(0)}\left|\left(\xi\left(\frac{|y|}{\lambda}\right)-1\right)\varphi_{\delta,0}(x)\right|^{2^*_s}dx\\ &\le C\int_{B_{2\lambda}(0)}|\varphi_{\delta,z}(x)|^{2^*_s}dx\\ &\to 0 \text{ for } (\delta,z)\in\mathbb{R}^N\times\mathbb{R}^+,\end{aligned}$$

we obtain that

$$|v_\lambda|^{2^*_s}_{2^*_s} = |v_\lambda|^{2^*_s}_{2^*_s,\mathbb{R}^N} \to |\varphi_{\delta,z}|^{2^*_s}_{2^*_s,\mathbb{R}^N} \text{ as } \lambda\to 0\ \forall (z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+ \tag{3.13}$$

and, together with $V\in L^{\frac{N}{2s}}(\Omega)$, conclude that

$$\int_{\mathbb{R}^N} V(x)|v_\lambda|^2\mathrm{d}x \to \int_{\mathbb{R}^N} V(x)|\varphi_{\delta,z}|^2\mathrm{d}x \text{ as } \lambda\to 0\ \forall (z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+. \tag{3.14}$$

Thanks to $V\in L^{\frac{N}{2s}}(\Omega)$, arguing as in the proof of Lemma 4.3 of [20], for any $\varepsilon>0$, there exist $\delta_1=\delta_1(\varepsilon)$ and $\delta_2=\delta_2(\varepsilon)$ such that

$$\int_{\mathbb{R}^N} V(x)|\varphi_{\delta,z}|^2\mathrm{d}x<\varepsilon$$

for $z\in\mathbb{R}^N$ and $\delta\in(0,\delta_1]\cup[\delta_2,+\infty)$. And then, combined with (3.14), we obtain that, as $\lambda\to 0$,

$$\int_{\mathbb{R}^N} V(x)|v_\lambda|^2\mathrm{d}x \to 0 \text{ as } \delta\to 0 \text{ or } \delta\to+\infty\ \forall z\in\mathbb{R}^N. \tag{3.15}$$

So it follows from (3.11), (3.12), (3.15), $\mathcal{J}_\infty(\varphi_{\delta,z})=m_\infty$, and $\langle\mathcal{J}'_\infty(\varphi_{\delta,z}),\varphi_{\delta,z}\rangle=0$ that, as $\lambda\to 0$,

$$\mathcal{J}(v_\lambda)\to m_\infty \text{ and } \langle\mathcal{J}'(v_\lambda,v_\lambda\rangle\to 0 \text{ as } \delta\to 0 \text{ or } \delta\to+\infty \text{ for } z\in\mathbb{R}^N.$$

Then, similar to the proof as in Proposition 2.2, we can conclude that, as $\lambda\to 0$, $t_{v_\lambda}\to 1$, as $\delta\to 0$ or $\delta\to+\infty$ for $z\in\mathbb{R}^N$. Consequently, (c) follows directly from the definition of $\mathcal{K}_\lambda(z,\delta)$. ◻

LEMMA 3.2. *Let* $(V_1)-(V_2)$ *hold. Then there is* $\lambda^*\in(0,\frac{1}{8})$ *such that for any* $\lambda<\lambda^*$,

$$\sup_{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+}\mathcal{J}(\mathcal{K}_\lambda(z,\delta))<2^{\frac{4s-\mu}{N-\mu+2s}}m_\infty.$$

Proof. It follows from (3.11), (3.12), (3.14), and $\langle\mathcal{J}'(\mathcal{K}_\lambda(z,\delta)),\mathcal{K}_\lambda(z,\delta)\rangle=\langle\mathcal{J}'(t_{v_\lambda}v_\lambda),t_{v_\lambda}v_\lambda\rangle=0$ that

$$\begin{aligned}t_{v_\lambda}^{(2\cdot 2^*_{\mu,s}-2)} &= \frac{\|v_\lambda\|^2+\int_\Omega V(x)v_\lambda^2\mathrm{d}x}{\int_\Omega(I_\mu*|v_\lambda|^{2^*_{\mu,s}})|v_\lambda|^{2^*_{\mu,s}}\mathrm{d}x} \le \frac{\|v_\lambda\|^2+|V|_{\frac{N}{2s}}|v_\lambda|^2_{2^*_s}}{\int_\Omega(I_\mu*|v_\lambda|^{2^*_{\mu,s}})|v_\lambda|^{2^*_{\mu,s}}\mathrm{d}x}\\ &\to \frac{\|\varphi_{\delta,z}\|^2_{\mathbb{R}^N}+|V|_{\frac{N}{2s}}|\varphi_{\delta,z}|^{2^*_s}_{2^*_s,\mathbb{R}^N}}{\int_{\mathbb{R}^N}(I_\mu*|\varphi_{\delta,z}|^{2^*_{\mu,s}})|\varphi_{\delta,z}|^{2^*_{\mu,s}}\mathrm{d}x} \text{ as } \lambda\to 0\ \forall(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+.\end{aligned}$$

Hence, by (V_1) and (V_2), we derive that

$$
\begin{aligned}
&\lim_{\lambda\to 0}\mathcal{J}(\mathcal{K}_\lambda(z,\delta))\\
&=\lim_{\lambda\to 0}\left(\frac{t_{v_\lambda}^2}{2}\|v_\lambda\|^2+\frac{t_{v_\lambda}^2}{2}\int_\Omega V(x)v_\lambda^2\mathrm{d}x-\frac{t_{v_\lambda}^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\int_\Omega (I_\mu*|v_\lambda|^{2^*_{\mu,s}})|v_\lambda|^{2^*_{\mu,s}}\mathrm{d}x\right)\\
&=\lim_{\lambda\to 0}\left(\frac{1}{2}-\frac{1}{2\cdot 2^*_{\mu,s}}\right)t_{v_\lambda}^{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu*|v_\lambda|^{2^*_{\mu,s}})|v_\lambda|^{2^*_{\mu,s}}\mathrm{d}x\\
&\le\frac{N-\mu+2s}{2(2N-\mu)}\left(\frac{\|\varphi_{\delta,z}\|^2_{\mathbb{R}^N}(\frac{1}{S_s}|V|_{\frac{N}{2s}}+1)}{\int_{\mathbb{R}^N}(I_\mu*|\varphi_{\delta,z}|^{2^*_{\mu,s}})|\varphi_{\delta,z}|^{2^*_{\mu,s}}\mathrm{d}x}\right)^{\frac{2\cdot 2^*_{\mu,s}}{2\cdot 2^*_{\mu,s}-2}}\int_{\mathbb{R}^N}(I_\mu*|\varphi_{\delta,z}|^{2^*_{\mu,s}})|\varphi_{\delta,z}|^{2^*_{\mu,s}}\mathrm{d}x\\
&\le\frac{N-\mu+2s}{2(2N-\mu)}\left(\frac{1}{S_s}|V|_{\frac{N}{2s}}+1\right)^{\frac{2N-\mu}{N-\mu+2s}}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}\\
&<2^{\frac{4s-\mu}{N-\mu+2s}}\frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}\\
&=2^{\frac{4s-\mu}{N-\mu+2s}}m_\infty\ \forall(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+.
\end{aligned}
$$

Consequently, we conclude that there exists a constant $\lambda^*\in(0,\frac{1}{8})$ such that for any $\lambda<\lambda^*$,

$$\sup_{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+}\mathcal{J}(\mathcal{K}_\lambda(z,\delta))<2^{\frac{4s-\mu}{N-\mu+2s}}m_\infty. \qquad \square$$

In subsequent discussions, we always assume Ω fixed with $\mathrm{diam}\,(\mathbb{R}^N\setminus\Omega):=\sup\{|x-y|:x,y\in\mathbb{R}^N\setminus\Omega\}<\lambda^*$, where $\lambda^*\in(0,1/8)$ is the constant obtained in Lemma 3.2. Hence, for any $x_0\in\mathbb{R}^N\backslash\Omega$, $\mathbb{R}^N\backslash\Omega\subset B_{\lambda^*}(x_0)$. Thus, we have that

$$\lambda=\inf\{\tau:\mathbb{R}^N\backslash\Omega\subset\overline{B_\tau(0)}\}<\lambda^*<\frac{1}{8},\ \mathbb{R}^N\backslash\Omega\subset B_{\frac{1}{8}}(0).$$

Define $\chi_i:\mathbb{R}^+\to\mathbb{R}, i=1,2$, by

$$\chi_1(t)=\begin{cases}4, t\le\dfrac{1}{4},\\[2mm] \dfrac{1}{t}, t>\dfrac{1}{4},\end{cases}\quad\text{and}\ \chi_2(t)=\begin{cases}0, t<1,\\ 1, t\ge 1.\end{cases}$$

Define a barycenter-type map $\alpha:D^{s,2}(\mathbb{R}^N)\to\mathbb{R}^N$ and a functional $\beta:D^{s,2}(\mathbb{R}^N)\to\mathbb{R}$ as

$$
\begin{aligned}
\alpha(u)&=\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N}\chi_1(|x|)x|(-\Delta)^{-\frac{s}{2}}u|^2\mathrm{d}x,\\
\beta(u)&=\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N}\chi_2(|x|)|(-\Delta)^{-\frac{s}{2}}u|^2\mathrm{d}x.
\end{aligned}
$$

Let

$$\mathcal{M}:=\left\{u\in\mathcal{N}:(\alpha(u),\beta(u))=\left(0,\frac{1}{2}\right)\right\}\subset D_0^{s,2}(\Omega).$$

LEMMA 3.3. *If $|z|\ge\frac{1}{2}$, then we have*

$$\alpha(\varphi_{\delta,z})=\frac{z}{|z|}+o(1)\ \mathit{as}\ \delta\to 0.$$

Proof. Fix $|z|\ge \frac{1}{2}$. We have $B_{\frac{1}{4}}(0)\cap B_\varepsilon(z)=\emptyset$ for any $\varepsilon>0$ small enough. Then, by the definition of $\varphi_{\delta,z}$ and thanks to Proposition 2.2 of [45], we have that

$$\begin{aligned}\left|\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{B_{\frac{1}{4}}(0)}4x|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\right| &\le \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{B_{\frac{1}{4}}(0)}4|x||(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\\ &\le C\int_{B_{\frac{1}{4}}(0)}|\nabla\varphi_{\delta,z}|^2\mathrm{d}x\\ &\le C\delta^{\frac{N-2s}{2}}|z|^2\to 0 \text{ as } \delta\to 0\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\left|\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathcal{A}}\frac{x}{|x|}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\right| &\le \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_{\frac{1}{4}}(0)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\\ &\le C_1\int_{\mathbb{R}^N\setminus B_{\frac{1}{4}}(0)}|\nabla\varphi_{\delta,z}|^2\mathrm{d}x\\ &\le C_2\delta^{N-2s}\to 0 \text{ as } \delta\to 0,\end{aligned} \tag{3.17}$$

where $\mathcal{A}=\mathbb{R}^N\setminus(B_\varepsilon(z)\cup B_{\frac{1}{4}}(0))$.

For any $x\in B_\varepsilon(z)$, considering $|z|\ge\frac{1}{2}$ and $\varepsilon>0$ small enough, we have

$$\left|\frac{x}{|x|}-\frac{z}{|z|}\right|<C_3\varepsilon.$$

Therefore, we deduce that

$$\begin{aligned}&\left|\frac{z}{|z|}-\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{B_\varepsilon(z)}\frac{x}{|x|}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\right|\\ &=\left|\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{B_\varepsilon(z)}\left(\frac{z}{|z|}-\frac{x}{|x|}\right)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\right.\\ &\qquad\left.+\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_\varepsilon(z)}\frac{z}{|z|}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\right|\\ &\le\frac{C_3\varepsilon}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\\ &\qquad+\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_\varepsilon(z)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\to 0 \text{ as } \delta\to 0.\end{aligned} \tag{3.18}$$

Hence, combining with (3.16), (3.17), and (3.18), we obtain that

$$\alpha(\varphi_{\delta,z})=\frac{z}{|z|}+o(1) \text{ as } \delta\to 0.$$

∎

LEMMA 3.4. *Suppose that* (V_1) *holds with* $|V|_{\frac{N}{2s}} \neq 0$. *Then we have*

$$c_0 := \inf_{u \in \mathcal{M} \cap \mathcal{P}} \mathcal{J}(u) > m_\infty, \tag{3.19}$$

where $\mathcal{P}$ *is the cone of nonnegative functions of* $D_0^{s,2}(\Omega)$. *Furthermore, as* $\lambda \to 0$, *there exist* $T > \frac{1}{2}, 0 < \delta_1 < \frac{1}{2}, \delta_2 > \frac{1}{2}$ *such that*

$$\begin{aligned}
&\text{(a) } \beta(\mathcal{K}_\lambda(z,\delta)) < \frac{1}{2}, |z| < \frac{1}{2} \text{ and } \delta \le \delta_1,\\
&\text{(b) } \left|\alpha(\mathcal{K}_\lambda(z,\delta)) - \frac{z}{|z|}\right| < \frac{1}{4}, |z| \ge \frac{1}{2} \text{ and } \delta \le \delta_1,\\
&\text{(c) } \beta(\mathcal{K}_\lambda(z,\delta)) > \frac{1}{2}, z \in \mathbb{R}^N \text{ and } \delta \ge \delta_2,\\
&\text{(d) } \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < \frac{c_0 + m_\infty}{2}, z \in \mathbb{R}^N \text{ and } \delta = \delta_1 \text{ or } \delta = \delta_2,\\
&\text{(e) } \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \in \left(m_\infty, \frac{c_0 + m_\infty}{2}\right), |z| \ge T \text{ and } \delta \in [\delta_1, \delta_2],\\
&\text{(f) } \Big(\alpha(\mathcal{K}_\lambda(z,\delta)), z\Big)_{\mathbb{R}^N} > 0, |z| = T \text{ and } \delta \in [\delta_1, \delta_2].
\end{aligned} \tag{3.20}$$

Proof. First, we prove (3.19). Obviously, $c_0 \ge m_\infty$, so to obtain (3.19), we suppose that $c_0 = m_\infty$ by contradiction. Hence, there is $\{u_n\} \subset \mathcal{M} \cap \mathcal{P}$ such that

$$\lim_{n\to\infty} \mathcal{J}(u_n) = m_\infty,\ \langle \mathcal{J}'(u_n), u_n\rangle = 0,\ \alpha(u_n) = 0,\ \beta(u_n) = \frac{1}{2}. \tag{3.21}$$

Thanks to Proposition 2.2, $\{u_n\}$ is not relatively compact. Then it follows from Corollary 2.1 that

$$u_n(x) = \varphi_{\delta_n, z_n}(x) + w_n(x), x \in \mathbb{R}^N,$$

where $\{z_n\} \in \mathbb{R}^N, \{\delta_n\} \in \mathbb{R}^+$, $\{w_n\} \subset D^{s,2}(\mathbb{R}^N)$ with $w_n \to 0$ in $D^{s,2}(\mathbb{R}^N)$, and φ_{δ_n,z_n} is a positive ground state solution of (2.8) realizing m_∞. In a subsequence sense, for (δ_n, z_n), one of the following conditions holds:

$$\begin{aligned}
&(1)\ \delta_n \to +\infty \text{ as } n \to +\infty,\ (2)\ \delta_n \to \delta \neq 0 \text{ as } n \to +\infty,\\
&(3)\ \delta_n \to 0 \text{ and } z_n \to z \text{ with } |z| < \frac{1}{2} \text{ as } n \to +\infty,\\
&(4)\ \delta_n \to 0\ \text{ as } n \to +\infty \text{ and } |z_n| \ge \frac{1}{2} \text{ for } n \text{ large.}
\end{aligned} \tag{3.22}$$

By definitions of $\alpha(u)$ and $\beta(u)$, thanks to (3.21), we have that

$$\alpha(\varphi_{\delta_n,z_n}) \to 0,\ \beta(\varphi_{\delta_n,z_n}) \to \frac{1}{2} \text{ as } n \to +\infty. \tag{3.23}$$

If $\delta_n \to +\infty$ as $n \to +\infty$, then we obtain that

$$\begin{aligned}
\beta(\varphi_{\delta_n,z_n}) &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N} \chi_2(|x|) |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta_n,z_n}|^2 \mathrm{d}x\\
&= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N \setminus B_1(0)} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta_n,z_n}|^2 \mathrm{d}x\\
&= 1 - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_1(0)} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta_n,z_n}|^2 \mathrm{d}x\\
&= 1 + o_n(1) \text{ as } n \to +\infty,
\end{aligned}$$

which is a contradiction.

If $\delta_n \to \delta \neq 0$ as $n \to +\infty$, then, by Proposition 2.2, we can obtain that $z_n \to +\infty$ as $n \to +\infty$. Then we have

$$\begin{aligned}\beta(\varphi_{\delta_n,z_n}) &= \beta(\varphi_{\delta,z_n}) + o_n(1)\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N}\chi_2(|x+z_n|)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2\mathrm{d}x + o_n(1)\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(-z_n)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2\mathrm{d}x + o_n(1)\\ &= 1 - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{B_1(-z_n)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2\mathrm{d}x + o_n(1)\\ &= 1 + o_n(1) \text{ as } n \to +\infty.\end{aligned}$$

Thanks to (3.23), we get a contradiction.

If $\delta_n \to 0$ and $z_n \to z$ with $|z| < \frac{1}{2}$ as $n \to +\infty$, then

$$\begin{aligned}\beta(\varphi_{\delta_n,z_n}) &= \beta(\varphi_{\delta_n,z}) + o_n(1)\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N}\chi_2(|x+z|)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,0}|^2\mathrm{d}x + o_n(1)\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(-z)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,0}|^2\mathrm{d}x + o_n(1)\\ &= o_n(1) \text{ as } n \to +\infty,\end{aligned}$$

which contradicts with (3.23).

If $\delta_n \to 0$ as $n \to +\infty$ and $|z_n| \geq \frac{1}{2}$ for n large, then, according to Lemma 3.3, we get

$$\alpha(\varphi_{\delta_n,z_n}) = \frac{z_n}{|z_n|} + o_n(1) \text{ as } n \to +\infty.$$

Obviously, it is impossible due to (3.23).

Consequently, it follows from the above discussions that $c_0 > m_\infty$.

In the following, we give the proof of (3.20).

By (c) of Lemma 3.1, as $\lambda \to 0$, one has

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) = \mathcal{J}(t_{v_\lambda}v_\lambda)) \to m_\infty \text{ as } \delta \to 0 \text{ uniformly in } z \in \mathbb{R}^N.$$

Then it follows from $\mathcal{K}_\lambda(z,\delta) \in \mathcal{N}$ and Lemma 3.3 that, as $\lambda \to 0$,

$$\beta(\mathcal{K}_\lambda(z,\delta)) \to \beta(\varphi_{\delta,z}) \text{ as } \delta \to 0 \text{ uniformly in } z \in \mathbb{R}^N. \tag{3.24}$$

Hence, we conclude that

$$\begin{aligned}\beta(\varphi_{\delta,z}) &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(0)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(-z)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2\mathrm{d}x\\ &= o(1) \text{ as } \delta \to 0 \text{ for } |z| < \frac{1}{2},\end{aligned}$$

which shows that, as $\lambda \to 0$, there is $0<\delta_1<\frac{1}{2}$ such that (a) holds.

Similarly, it follows from (c) of Lemma 3.1 and Corollary 2.1 that, as $\lambda \to 0$,

$$\alpha(\mathcal{K}_\lambda(z,\delta)) \to \alpha(\varphi_{\delta,z}) \text{ as } \delta \to 0 \text{ uniformly in } z \in \mathbb{R}^N.$$

Then, according to Lemma 3.3, as $\lambda \to 0$, there is $0<\delta_1<\frac{1}{2}$ such that (b) holds.

By (c) of Lemma 3.1 and Corollary 2.1, as $\lambda \to 0$, we also obtain that

$$\beta(\mathcal{K}_\lambda(z,\delta)) \to \beta(\varphi_{\delta,z}) \text{ as } \delta \to 0 \text{ uniformly in } z \in \mathbb{R}^N. \tag{3.25}$$

Thanks to

$$\int_{B_1(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \le C\delta^{-\frac{N}{2}-1+s} \to 0 \text{ as } \delta \to +\infty,$$

we have that

$$\begin{aligned}\beta(\varphi_{\delta,z}) &= 1 - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_1(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \\ &= 1 + o(1) \text{ as } \delta \to +\infty.\end{aligned}$$

Hence, as $\lambda \to 0$, it follows from (3.25) that there is $\delta_2 > \frac{1}{2}$ such that (c) holds.

According to (c) of Lemma 3.1, as $\lambda \to 0$, we have

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) = \mathcal{J}(t_{v_\lambda} v_\lambda)) \to m_\infty \text{ as } \delta \to 0 \text{ or } \delta \to +\infty \text{ uniformly in } z \in \mathbb{R}^N.$$

Thanks to (3.19), as $\lambda \to 0$, there exist $0<\delta_1<\frac{1}{2}<\delta_2$ such that (d) holds.

By $\mathcal{K}_\lambda(z,\delta) \in \mathcal{N}$, we have $\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \ge m$. Hence, thanks to Proposition 2.2, we obtain that

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) > m = m_\infty \forall z \in \mathbb{R}^N, \delta > 0.$$

By (b) of Lemma 3.1, one has

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) = \mathcal{J}(t_{v_\lambda} v_\lambda)) \to m_\infty \text{ as } |z| \to +\infty$$

uniformly for every bounded λ and bounded δ away from 0. Thanks to (3.19), there is $T_1 > \frac{1}{2}$ satisfying

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < \frac{c_0 + m_\infty}{2}, |z| \ge T_1, \delta_1 \le \delta \le \delta_2, \lambda < 1.$$

That is, (e) holds.

According to (b) of Lemma 3.1, Corollary 2.1, and the fact that $\mathcal{K}_\lambda(z,\delta) \in \mathcal{N}$, we have that

$$\Big(\alpha(\mathcal{K}_\lambda(z,\delta)), z\Big)_{\mathbb{R}^N} \to \Big(\alpha(\varphi_{\delta,z}), z\Big)_{\mathbb{R}^N} \text{ as } |z| \to +\infty \tag{3.26}$$

uniformly in $\delta_1 \le \delta \le \delta_2, \lambda < 1$.

Let

$$(\mathbb{R}^N)_z^+ := \{x \in \mathbb{R}^N : (x,z)_{\mathbb{R}^N} > 0\} \text{ and } (\mathbb{R}^N)_z^- := \{x \in \mathbb{R}^N : (x,z)_{\mathbb{R}^N} \le 0\}.$$

Since $\delta \in [\delta_1, \delta_2]$, there exist a large $T_2 > 0$ and $r \in (0, 1/4)$ such that if $|z| \geq T_2$, then the ball $B_r(\widetilde{z}) \subset (\mathbb{R}^N)^+_z$ with $\widetilde{z}$ satisfying $|z - \widetilde{z}| = \frac{1}{2}$, and by the definition of $\varphi_{\delta,z}$, one has

$$|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \geq C_0 > 0, x \in B_r(\widetilde{z}).$$

Hence, for any $|z| \geq T_2$, $\delta \in [\delta_1, \delta_2]$, one has

$$\begin{aligned}\Big(\alpha(\varphi_{\delta,z}), z\Big)_{\mathbb{R}^N} &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)^+_z} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N}\mathrm{d}x \\ &\quad + \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)^-_z} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N}\mathrm{d}x \\ &:= I_1 + I_2.\end{aligned}$$

Moreover, we have

$$\begin{aligned} I_1 &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)^+_z} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N}\mathrm{d}x \\ &\geq \frac{|z|}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_r(\widetilde{z})} \frac{C_0(x,z)_{\mathbb{R}^N}}{|x||z|}\mathrm{d}x \\ &\geq C_1|z| \int_{B_r(\widetilde{z})} \frac{1}{|x|}\mathrm{d}x \\ &= C_2|z|.\end{aligned}$$

Arguing as Lemma 4.7 of [20], choose $T_3 > 0$ large enough such that $z \in \mathbb{R}^N$ with $|z| > T_3$. We have

$$\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)^-_z} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x < \frac{C_2}{2}.$$

Then we have

$$\begin{aligned} I_2 &\geq \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)^-_z} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N}\mathrm{d}x \\ &\geq -\frac{|z|}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)^-_z} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \\ &> -\frac{C_2|z|}{2}.\end{aligned}$$

Therefore,

$$\Big(\alpha(\varphi_{\delta,z}), z\Big)_{\mathbb{R}^N} = I_1 + I_2 \geq \frac{C_2}{2}|z| > 0$$

for all $z \in \mathbb{R}^N$ with $|z| > T = \max\{T_1, T_2, T_3\} > 0$ and for all $\delta_1 \leq \delta \leq \delta_2$. ◻

Let

$$\Pi_{\delta,z} := \Big\{(z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+ : |z| < T, \delta \in [\delta_1, \delta_1]\Big\},$$

where δ_1, δ_2, and T are constants defined as in Lemma 3.4. Denote

$$\partial\Pi_{\delta,z} := \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4,$$

where

$$\Pi_1 = \left\{(z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+ : |z| < \frac{1}{2}, \delta = \delta_1\right\},$$
$$\Pi_2 = \left\{(z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+ : \frac{1}{2} \leq |z| < T, \delta = \delta_1\right\},$$
$$\Pi_3 = \left\{(z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+ : |z| \leq T, \delta = \delta_2\right\},$$
$$\Pi_4 = \left\{(z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+ : |z| = T, \delta \in [\delta_1, \delta_1]\right\}.$$

Define $\Theta \subset D_0^{s,2}(\Omega)$ by

$$\Theta := \left\{\mathcal{K}_\lambda(z,\delta) : (z,\delta) \in \Pi_{\delta,z}\right\}.$$

It is easy to see that $\Theta \subset \mathcal{P} \cap \mathcal{N}$. Set

$$\mathcal{E} := \left\{\gamma : \gamma \in C(\mathcal{P} \cap \mathcal{N}, \mathcal{P} \cap \mathcal{N}), \gamma(u) = u \text{ for any } u \text{ with } \mathcal{J}(u) < \frac{c_0 + m_\infty}{2}\right\},$$

$$\mathcal{F} := \left\{D \subset \mathcal{P} \cap \mathcal{N} : D = \gamma(\Theta), \gamma \in \mathcal{E}\right\}.$$

LEMMA 3.5. *Let* (V_1) *hold and* $D \in \mathcal{F}$. *Then, as* $\lambda \to 0$, *we have*

$$D \cap \mathcal{M} \neq \emptyset.$$

Proof. To prove this lemma, we just prove that for any $\gamma \in \mathcal{E}$, as $\lambda \to 0$, there is $(z^*, \delta^*) \in \Pi_{\delta,z}$ so that

$$((\alpha,\beta) \circ \gamma \circ \mathcal{K}_\lambda)(z^*, \delta^*) = \left(0, \frac{1}{2}\right). \tag{3.27}$$

Define $\mathcal{L}_\gamma : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^N \times \mathbb{R}^+$ and $\mathcal{L} : \Pi_{\delta,z} \to \mathbb{R}^N \times \mathbb{R}^+$ by

$$\mathcal{L}_\gamma = (\alpha,\beta) \circ \gamma \circ \mathcal{K}_\lambda, \;\; \mathcal{L} = (\alpha,\beta) \circ \mathcal{K}_\lambda.$$

First, we prove that, as $\lambda \to 0$,

$$deg\left(\mathcal{L}_\gamma, \Pi_{\delta,z}, \left(0, \frac{1}{2}\right)\right) = deg\left(\mathcal{L}, \Pi_{\delta,z}, \left(0, \frac{1}{2}\right)\right). \tag{3.28}$$

It follows from (d) and (e) of (3.20) that, as $\lambda \to 0$,

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < \frac{c_0 + m_\infty}{2}, (z,\delta) \in \partial\Pi_{\delta,z}.$$

Thus, as $\lambda \to 0$, we have

$$\gamma(\mathcal{K}_\lambda(z,\delta)) = \mathcal{K}_\lambda(z,\delta), (z,\delta) \in \partial\Pi_{\delta,z}.$$

So we obtain that

$$\mathcal{L}_\gamma(z,\delta) = ((\alpha,\beta)\circ\mathcal{K}_\lambda)(z,\delta) = \mathcal{L}(z,\delta), (z,\delta) \in \partial\Pi_{\delta,z},$$

which concludes that (3.28) holds.

Second, we need to prove that, as $\lambda \to 0$,

$$deg\left(\mathcal{L}, \Pi_{\delta,z}, \left(0, \frac{1}{2}\right)\right) = 1. \tag{3.29}$$

Let

$$\Upsilon(z,\delta,t) = t\mathcal{L}(z,\delta) + (1-t)(z,\delta), t \in [0,1].$$

It follows from the homotopy invariance property of the topological degree and the fact that

$$deg\left(Id_{\Pi_{\delta,z}}, \Pi_{\delta,z}, \left(0, \frac{1}{2}\right)\right) = 1$$

that to get (3.29), we just prove that, as $\lambda \to 0$,

$$\Upsilon(z,\delta,t) \neq \left(0, \frac{1}{2}\right) \forall (z,\delta) \in \partial\Pi_{\delta,z} \text{ and } t \in [0,1]. \tag{3.30}$$

If $(z,\delta) \in \Pi_1$, then, thanks to $\delta_1 < \frac{1}{2}$ and (a) of (3.20), we have

$$t(\beta\circ\mathcal{K}_\lambda)(z,\delta_1) + (1-t)\delta_1 < \frac{1}{2},\ t \in [0,1]. \tag{3.31}$$

If $(z,\delta) \in \Pi_2$, then, since $\delta_1 < \frac{1}{2}$, it follows from (b) of (3.20) that

$$\left|\alpha(\mathcal{K}_\lambda(z,\delta)) - \frac{z}{|z|}\right| < \frac{1}{4},$$

which concludes that

$$\begin{aligned} |(1-t)z + t(\alpha\circ\mathcal{K}_\lambda)(z,\delta_1)| &\geq \left|(1-t)z + t\frac{z}{|z|}\right| - \left|t(\alpha\circ\mathcal{K}_\lambda)(z,\delta_1) - t\frac{z}{|z|}\right| \\ &\geq (1-t)|z| + t - \frac{t}{4} \\ &\geq \frac{1}{2} + \frac{t}{4} > 0,\ t \in [0,1]. \end{aligned} \tag{3.32}$$

If $(z,\delta) \in \Pi_3$, then, by $\delta_2 > \frac{1}{2}$ and (c) of (3.20), we deduce that

$$t(\beta\circ\mathcal{K}_\lambda)(z,\delta_2) + (1-t)\delta_2 > (1-t)\frac{1}{2} + \frac{t}{2} = \frac{1}{2},\ t \in [0,1]. \tag{3.33}$$

If $(z,\delta) \in \Pi_4$, then, according to (f) of (3.20), we have

$$\Big(t(\alpha\circ\mathcal{K}_\lambda)(z,\delta) + (1-t)z, z\Big)_{\mathbb{R}^N} = (1-t)|z|^2 + t\Big((\alpha\circ\mathcal{K}_\lambda)(z,\delta), z\Big)_{\mathbb{R}^N} > 0, t \in [0,1]. \tag{3.34}$$

Consequently, by (3.31)–(3.34), we get (3.30), and then (3.29) holds. Therefore, combining (3.28) with (3.29), we obtain that $D \cap \mathcal{M} \neq \emptyset$. ∎

4. The proof of main results.

Proof. Let

$$c^* := \inf_{D\in\mathcal{F}} \sup_{u\in D} \mathcal{J}(u),$$
$$K_{c^*} := \{u \in \mathcal{P}\cap\mathcal{N} : \mathcal{J}(u) = c^*,\ \mathcal{J}'(u) = 0\},$$
$$\mathcal{J}^r := \{u \in \mathcal{N} : \mathcal{J}(u) \le r\},\ r \in \mathbb{R}.$$

Let λ^* be small enough such that Lemmas 3.2 and 3.5 hold for any $\lambda < \lambda^*$. Fix λ with $\lambda < \lambda^*$. To prove Theorem 1.1, we only prove that $K_{c^*} \neq \emptyset$. Suppose by contradiction that $K_{c^*} = \emptyset$. It follows from Lemmas 3.4 and 3.5 that

$$c^* \ge \inf_{u\in\mathcal{M}\cap\mathcal{P}} \mathcal{J}(u) = c_0 > m_\infty.$$

By Lemma 3.2, we obtain that $c^* < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty$ due to $\Theta \subset \mathcal{F}$. Therefore, we have

$$m_\infty < c^* < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty.$$

According to Corollary 2.2, $\mathcal{J}$ satisfies Palais–Smale condition in

$$\mathcal{P}\cap\mathcal{N}\cap\{u\in D_0^{s,2}(\Omega) : m_\infty < \mathcal{J}(u) < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty\}.$$

So it follows from a variant due to Hofer [22] of the classical deformation lemma (see [50, 53]) that there is a continuous map

$$\psi : [0,1]\times\mathcal{P}\cap\mathcal{N} \to \mathcal{P}\cap\mathcal{N}$$

and $\varepsilon_0 > 0$ satisfying

(a) $\mathcal{J}^{c^*+\varepsilon_0}\backslash\mathcal{J}^{c^*-\varepsilon_0} \subset\subset \mathcal{J}^{2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty}\backslash\mathcal{J}^{\frac{c_0+m_\infty}{2}}$;
(b) $\psi(0,u) = u$;
(c) $\psi(t,u) = u, u \in \mathcal{J}^{c^*-\varepsilon_0} \cup \{\mathcal{P}\cap\mathcal{N}\backslash\mathcal{J}^{c^*+\varepsilon_0}\}$, $t\in[0,1]$;
(d) $\psi(1, \mathcal{J}^{c^*+\frac{\varepsilon_0}{2}}) \subset \mathcal{J}^{c^*-\frac{\varepsilon_0}{2}}$.

By definition of c^*, there exists $D^* \in \mathcal{F}$ such that

$$c^* \le \sup_{u\in D^*} \mathcal{J}(u) < c^* + \frac{\varepsilon_0}{2}.$$

Hence, $\psi(1, D^*) \in \mathcal{F}$, and it follows from (d) that

$$c^* \le \sup_{u\in\psi(1,D^*)} \mathcal{J}(u) < c^* - \frac{\varepsilon_0}{2},$$

which is a contradiction. Thus, $K_{c^*} \neq \emptyset$, and Theorem 1.1 is proved. ∎

Conflict of interest. The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

Data availability. Data sharing is not applicable to this article, as no data sets were generated or analyzed during the current study.

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