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EXISTENCE OF A SIGN-CHANGING SOLUTION FOR MULTI-PHASE PROBLEMS VIA NEHARI METHOD

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To my dear father on the occasion of his 80th birthday

Abstract. We explore a Dirichlet problem driven by a multi-phase operator with three variable exponents. Here, under very general assumptions on the exponents and the nonlinearity, using variational tools and the Nehari method, we are able to produce a sign-changing solution for such problem.

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$ be a bounded domain whose boundary is Lipschitz. Also, let $m \in C(\overline{\Omega})$ such that $m(x) > 1$ for all $x \in \overline{\Omega}$. Then, we take

$$m^- = \min_{x\in\overline{\Omega}} m(x) \quad \text{and} \quad m^+ = \max_{x\in\overline{\Omega}} m(x).$$

Next, we denote by p, q, r and μ_1, μ_2 functions satisfying the following assumptions:

(H1) $p, q, r \in C(\overline{\Omega})$ are such that

$$1 < p^- \leq p(x) < N,$$
$$p(x) < q(x) < r(x) \leq r^+ < (p^*)^- \leq p^*(x) := \frac{Np(x)}{N - p(x)}$$

for all $x \in \overline{\Omega}$;
$\mu_1(\cdot),\, \mu_2(\cdot) \in L^\infty(\Omega) \setminus \{0\}$ are such that

$$\mu_1(\cdot),\, \mu_2(\cdot) \geq 0$$

for all $x \in \Omega$.

Thus, we focus on the Dirichlet problem

$$\begin{aligned} -\operatorname{div}\mathcal{A}(u) &= -|u|^{p(x)-2}u + f(x,u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

driven by the operator $\operatorname{div}\mathcal{A}$ that is the multi-phase operator with variable exponents defined by

$$\operatorname{div}\mathcal{A}(u) := \operatorname{div}\left[\left(|\nabla u|^{p(x)-2} + \mu_1(x)|\nabla u|^{q(x)-2} + \mu_2(x)|\nabla u|^{r(x)-2}\right)\nabla u\right] \tag{1.2}$$

for any function u belonging to an appropriate Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{T}}(\Omega)$, which will be introduced in Section 2. We point out that such an operator is a proper generalization of the double-phase operator with variable exponents.

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Further, differently from the double-phase case, we have to consider two potential changes of phase rather than only one. For this reason, the operator introduced in (1.2) is of relevant interest in mathematical models of physics phenomena. Also, we recall that functionals of type

$$u \mapsto \int_\Omega \left(|\nabla u|^{p(x)} + \mu_1(x)|\nabla u|^{q(x)} + \mu_2(x)|\nabla u|^{r(x)}\right) \mathrm{d}x,$$

where p, q are constant and $\mu_2 = 0$, were considered by Marcellini [13] and Zhikov [22] in the context of homogenization and elasticity, by Zhikov [23, 24] and Papageorgiou-Rădulescu-Repovš [15] in the study of duality theory and of the Lavrentiev gap phenomenon, and by Marcellini [12, 13] in the study of problems of the calculus of variations.

Now, the aim of our paper is in showing that, under very general assumptions on the exponents and the nonlinearity f (see hypotheses (H1), (H2), and (H3)), problem (1.1) admits at least one sign-changing solution in $W_0^{1,\mathcal{T}}(\Omega)$ (see Theorem 4.6). In order to establish our existence result, we make use of variational tools and of the Nehari method which permits us to produce sign-changing solutions via critical point theory. Precisely, here we consider the Nehari manifold $\mathcal{N}$ corresponding to the functional ϕ associated to problem (1.1) (see (4.2) and Section 4). We stress that $\mathcal{N}$ contains all the nontrivial weak solutions of problem (1.1), and it is much smaller than $W_0^{1,\mathcal{T}}(\Omega)$. For this reason, $\phi|_{\mathcal{N}}$ can have nice properties which fail to be true globally. Thus, following the basic idea used in [11] and then in [3, 9], we first derive several properties which characterize $\phi|_{\mathcal{N}}$ (see Propositions 4.2 - 4.5). Next, starting from such properties and using the quantitative deformation lemma (see Lemma 2.4), we are able to produce a sign-changing solution for problem (1.1) in $W_0^{1,\mathcal{T}}(\Omega)$. We point out that when $\mu_2 = 0$, problem (1.1) reduces to a double-phase problem. Consequently, our main result establishes the existence of a sign-changing solution for the double-phase case as well.

As we already said, the Nehari method is a very handy and powerful tool which permits to determine solutions via critical point theory. For this reason, there are several works concerning existence results obtained applying such method. We cite for example the papers of Alves-Hamidi [1] for quasilinear problems, Crespo-Blanco-Winkert [3] for superlinear double phase problems with variable exponents, Gasiński-Winkert [9] for a double phase problem with nonlinear boundary condition and constant exponents, Liu-Dai [11] for double phase problem with constant exponents, Papageorgiou-Repovš-Vetro [16] for singular double phase problems, and Wu [21] for concave-convex elliptic problems in $\mathbb{R}^N$. Now, problems driven by a multi-phase operator as defined in (1.2) were studied in [2, 4, 6, 17, 18, 19]. In particular, a multi-phase obstacle problem with a multivalued reaction term depending on the gradient of the solution was considered in [2]. Existence and uniqueness results for Dirichlet problems with a nonlinearity which has gradient dependence were established in [4]. The existence of positive and negative weak solutions for multi-phase problems with a parametric concave term in the reaction was studied in [6]. A multi-phase problem with nonlinearity that satisfies general structure conditions was analyzed in [17], where the existence of extremal constant sign solutions was proved. A Kirchhoff-type multi-phase problem with a reaction term which is defined only locally was considered in [18]. Instead, a whole sequence of nontrivial weak solutions to a multi-phase problem which diverges in $W_0^{1,\mathcal{T}}(\Omega)$ was produced in [19]. We emphasize that the mentioned results above concerning multi-phase problems do not establish the existence of sign-changing solutions. This is the first work which moves in such direction.

2. **Preliminaries.** In the analysis of problem (1.1), we will make use of the variable exponent Sobolev spaces and of the Musielak-Orlicz Sobolev spaces. For this reason, we here collect some basic facts about them. For a more detailed overview on these topics we refer to the books of Harjulehto-Hästö [10] and Musielak [14].

Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$ be a bounded domain whose boundary is Lipschitz. Given $m \in C(\overline{\Omega})$ such that $m(x) > 1$ for all $x \in \overline{\Omega}$, we here write $L^{m(\cdot)}(\Omega)$ to denote the variable exponent Lebesgue space, that is,

$$L^{m(\cdot)}(\Omega) = \left\{u \in M(\Omega) \,:\, \rho_{m(\cdot)}(u) < +\infty\right\}$$

where $M(\Omega)$ stands for the set of all measurable functions $u\colon \Omega \to \mathbb{R}$ and the modular $\rho_{m(\cdot)}$ is defined by

$$\rho_{m(\cdot)}(u) := \int_\Omega |u|^{m(x)}\,\mathrm{d}x.$$

As usual, on $L^{m(\cdot)}(\Omega)$, we consider the Luxemburg norm, that is, we put

$$\|u\|_{m(\cdot)} := \inf\left\{\alpha > 0 \,:\, \rho_{m(\cdot)}\left(\frac{u}{\alpha}\right) \leq 1\right\}$$

for all $u \in L^{m(\cdot)}(\Omega)$. With this norm, $L^{m(\cdot)}(\Omega)$ becomes a separable, uniformly convex, and hence reflexive Banach space, whose dual space is given by $L^{m'(\cdot)}(\Omega)$, with $m'(\cdot)$ the conjugate variable exponent to $m(\cdot)$, that is,

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

We recall that given $m_1, m_2 \in C(\overline{\Omega})$, with $1 < m_1(x) \leq m_2(x)$ for all $x \in \overline{\Omega}$, we have the continuous embedding

$$L^{m_2(\cdot)}(\Omega) \hookrightarrow L^{m_1(\cdot)}(\Omega).$$

Also, we point out that the norm $\|\cdot\|_{m(\cdot)}$ and the modular $\rho_{m(\cdot)}$ are related by the following relations.

Proposition 2.1. *Let $m \in C(\overline{\Omega})$ be such that $m(x) > 1$ for all $x \in \overline{\Omega}$. Then, the following hold:*

- (j) $\|u\|_{m(\cdot)} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{m(\cdot)}(u) < 1$ *(resp.* $> 1, = 1$*);*
- (jj) *if* $\|u\|_{m(\cdot)} < 1$ *then* $\|u\|_{m(\cdot)}^{m^+} \leq \rho_{m(\cdot)}(u) \leq \|u\|_{m(\cdot)}^{m^-}$*;*
- (jjj) *if* $\|u\|_{m(\cdot)} > 1$ *then* $\|u\|_{m(\cdot)}^{m^-} \leq \rho_{m(\cdot)}(u) \leq \|u\|_{m(\cdot)}^{m^+}$*;*
- (jv) $\|u\|_{m(\cdot)} \to 0$ *if and only if* $\rho_{m(\cdot)}(u) \to 0$*;*
- (v) $\|u\|_{m(\cdot)} \to +\infty$ *if and only if* $\rho_{m(\cdot)}(u) \to +\infty$*.*

Next, we denote by $W^{1,m(\cdot)}(\Omega)$ the variable exponent Sobolev space corresponding to $L^{m(\cdot)}(\Omega)$. Such space is defined by

$$W^{1,m(\cdot)}(\Omega) = \left\{u \in L^{m(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{m(\cdot)}(\Omega)\right\}.$$

We equip it with the natural norm

$$\|u\|_{1,m(\cdot)} := \|u\|_{m(\cdot)} + \|\nabla u\|_{m(\cdot)},$$

for all $u \in W^{1,m(\cdot)}(\Omega)$, where $\|\nabla u\|_{m(\cdot)} = \|\,|\nabla u|\,\|_{m(\cdot)}$. Then, we use $W_0^{1,m(\cdot)}(\Omega)$ in order to denote the completion of $C_0^\infty(\Omega)$ in $W^{1,m(\cdot)}(\Omega)$. We underline that

$W^{1,m(\cdot)}(\Omega)$ and $W_0^{1,m(\cdot)}(\Omega)$ are uniformly convex, separable, and reflexive Banach spaces.

Now, let $\mathcal{T}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ be the nonlinear function defined by

$$\mathcal{T}(x,t) = t^{p(x)} + \mu_1(x)t^{q(x)} + \mu_2(x)t^{r(x)}$$

for all $x \in \Omega$ and for all $t \geq 0$, where the exponents and the functions μ_i are as given in hypothesis (H1). We point out that $\mathcal{T}$ is a locally integrable, generalized N-function satisfying the Δ_2-condition (see Section 3 of [4]). Hence, we know that the Musielak-Orlicz space $L^{\mathcal{T}}(\Omega)$ is given by

$$L^{\mathcal{T}}(\Omega) = \{u \in M(\Omega) \,:\, \rho_{\mathcal{T}}(u) < +\infty\}$$

furnished with the Luxemburg norm

$$\|u\|_{\mathcal{T}} := \inf\left\{\beta > 0 \,:\, \rho_{\mathcal{T}}\left(\frac{u}{\beta}\right) \leq 1\right\}$$

for all $u \in L^{\mathcal{T}}(\Omega)$, where the modular $\rho_{\mathcal{T}}(\cdot)$ is defined by

$$\rho_{\mathcal{T}}(u) := \int_\Omega \mathcal{T}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu_1(x)|u|^{q(x)} + \mu_2(x)|u|^{r(x)}\right)\mathrm{d}x.$$

From Proposition 3.2 of [4], we see that the modular $\rho_{\mathcal{T}}$ and the norm $\|\cdot\|_{\mathcal{T}}$ are related as follows.

Proposition 2.2. *Let hypothesis (H1) be satisfied. Then, the following hold:*

- (j) $\|u\|_{\mathcal{T}} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{\mathcal{T}}(u) < 1$ *(resp.* $> 1, = 1$*);*
- (jj) *if* $\|u\|_{\mathcal{T}} < 1$ *then* $\|u\|_{\mathcal{T}}^{r^+} \leq \rho_{\mathcal{T}}(u) \leq \|u\|_{\mathcal{T}}^{p^-}$*;*
- (jjj) *if* $\|u\|_{\mathcal{T}} > 1$ *then* $\|u\|_{\mathcal{T}}^{p^-} \leq \rho_{\mathcal{T}}(u) \leq \|u\|_{\mathcal{T}}^{r^+}$*;*
- (jv) $\|u\|_{\mathcal{T}} \to 0$ *if and only if* $\rho_{\mathcal{T}}(u) \to 0$*;*
- (v) $\|u\|_{\mathcal{T}} \to +\infty$ *if and only if* $\rho_{\mathcal{T}}(u) \to +\infty$*.*

The Musielak-Orlicz Sobolev space corresponding to $L^{\mathcal{T}}(\Omega)$ is

$$W^{1,\mathcal{T}}(\Omega) = \left\{u \in L^{\mathcal{T}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{T}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{T}} := \|u\|_{\mathcal{T}} + \|\nabla u\|_{\mathcal{T}},$$

for all $u \in W^{1,\mathcal{T}}(\Omega)$, where $\|\nabla u\|_{\mathcal{T}} := \|\,|\nabla u|\,\|_{\mathcal{T}}$. We use $W_0^{1,\mathcal{T}}(\Omega)$ in order to denote the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{T}}(\Omega)$. Now, according to Proposition 3.1 of [4], we have that the spaces $L^{\mathcal{T}}(\Omega)$, $W^{1,\mathcal{T}}(\Omega)$, and $W_0^{1,\mathcal{T}}(\Omega)$ are reflexive Banach spaces. In addition, from Proposition 3.3 of [4], we see that the classical Sobolev embedding results extend to them in the following way.

Proposition 2.3. *Let hypothesis (H1) be satisfied. Then, the following hold:*

- (j) $L^{\mathcal{T}}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$, $W^{1,\mathcal{T}}(\Omega) \hookrightarrow W^{1,m(\cdot)}(\Omega)$, $W_0^{1,\mathcal{T}}(\Omega) \hookrightarrow W_0^{1,m(\cdot)}(\Omega)$ *are continuous for all* $m \in C(\overline{\Omega})$ *with* $1 \leq m(x) \leq p(x)$ *for all* $x \in \overline{\Omega}$*;*
- (jj) $W^{1,\mathcal{T}}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{T}}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ *are compact for all* $m \in C(\overline{\Omega})$ *with* $1 \leq m(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$.

Also, we point out that it is possible to endow the space $W_0^{1,\mathcal{T}}(\Omega)$ with the equivalent norm given by

$$\|u\| := \|\nabla u\|_{\mathcal{T}} \quad \text{for all } u \in W_0^{1,\mathcal{T}}(\Omega),$$

see Proposition 3.4 (ii) of [4].

Finally, we remark that the nonlinear operator which sends

$$u \in W_0^{1,\mathcal{T}}(\Omega) \quad \text{into} \quad \int_\Omega \mathcal{A}(u) \cdot \nabla(\cdot)\,\mathrm{d}x \in (W_0^{1,\mathcal{T}}(\Omega))^*,$$

where $\mathcal{A}(u)$ is as given in (1.2), has several notable properties. In particular, Proposition 4.5 of [4] guarantees that it is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone, and in addition satisfies the (S_+)-property, that is,

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\mathcal{T}}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \int_\Omega \mathcal{A}(u) \cdot \nabla(u_n - u)\,\mathrm{d}x \le 0$$

imply $u_n \to u \quad \text{in} \quad W_0^{1,\mathcal{T}}(\Omega)$.

We conclude this section recalling the quantitative deformation lemma which will be a fundamental tool in order to produce the main result of this paper. The following version of lemma may be found in Willem, see [20, Lemma 2.3].

Lemma 2.4 (Quantitative deformation lemma)**.** *Let X be a Banach space and $\phi \in C^1(X,\mathbb{R})$. Also, let S be a nonempty subset of X, $a_0 \in \mathbb{R}$, and $\delta, \epsilon > 0$ such that*

$$\textit{for all } u \in \phi^{-1}([a_0 - 2\epsilon, a_0 + 2\epsilon]) \cap S_{2\delta} \quad \textit{it results} \quad \|\phi'(u)\|_* \ge \frac{8\epsilon}{\delta}$$

where

$$S_{2\delta} = \{u \in X : d(u,S) := \inf_{v\in S} \|u - v\| < 2\delta\}.$$

Then, there exists $\eta \in C([0,1] \times X, X)$ satisfying the following conditions:
- (j) *$\eta(t,u) = u$ if $t = 0$ or $u \notin \phi^{-1}([a_0 - 2\epsilon, a_0 + 2\epsilon]) \cap S_{2\delta}$;*
- (jj) *$\phi(\eta(1,u)) \le a_0 - \epsilon$ for all $u \in \phi^{-1}((-\infty, a_0 + \epsilon]) \cap S$;*
- (jjj) *$\eta(t,\cdot)$ is a homeomorphism of X for all $t \in [0,1]$;*
- (jv) *$\|\eta(t,u) - u\| \le \delta$ for all $u \in X$ and $t \in [0,1]$;*
- (v) *$\phi(\eta(\cdot,u))$ is decreasing for all $u \in X$;*
- (vj) *$\phi(\eta(t,u)) < a_0$ for all $u \in \phi^{-1}((-\infty, a_0]) \cap S_\delta$ and $t \in (0,1]$, being $S_\delta = \{u \in X : \inf_{v\in S} \|u - v\| < \delta\}$.*

Lastly, we fix some notation. For any $s \in \mathbb{R}$, we set $s_\pm = \max\{\pm s, 0\}$, which means $s = s^+ - s^-$. Thus, for any function $u\colon \Omega \to \mathbb{R}$, we write $u_\pm(\cdot) = [u(\cdot)]_\pm$. Also, given a Banach space X with dual space X^*, we use $\langle\cdot,\cdot\rangle$ in order to denote the duality pairing between X and X^*. With the goal to streamline the notation, in the next sections we will use C and $\tilde{C}$ in order to denote positive constants, which may change from line to line, but do not depend on the crucial quantities.

3. **Hypotheses.** In this section, we formulate our assumptions on the reaction term f. Also, we give an additional monotonicity condition on $p \in C(\overline{\Omega})$ which is needed in order to establish the main result of the paper. We start by the hypotheses on the nonlinearity f.

(H2) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:
- (i) there exist $l \in C(\overline{\Omega})$, with $r^+ < l(x) \le l^+ < (p^*)^-$, and $b_0 > 0$ such that

$$|f(x,s)| \le b_0\,(1 + |s|^{l(x)-1})$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$;

(ii)

$$\lim_{s\to\pm\infty} \frac{f(x,s)}{|s|^{r^+-2}s} = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii)

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p(x)-2}s} = 0 \quad \text{uniformly for a.a. } x \in \Omega;$$

(iv)

$$f(x,ts)\, s \,<\, t^{r^+-1} f(x,s)\, s$$

for a.a. $x \in \Omega$, for all $t \in (0,1)$ and for all $s \in \mathbb{R} \setminus \{0\}$.

Remark 3.1. We point out that from the continuity of $f(x,\cdot)$ along with hypothesis (H2)(iii), it follows that

$$f(x,0) = 0 \quad \text{for a.a. } x \in \Omega.$$

Example 3.2. Let $\theta \in (r^+, l^-)$. Then, the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,s) = 3\,|s|^{\theta-2}s \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R} \setminus \{0\}$$

satisfies all the assumptions in (H2).

Next, we claim that the following additional monotonicity condition on $p \in C(\overline{\Omega})$ holds.

(H3) There exists $\zeta_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$, the function $p_x \colon \Omega_x \to \mathbb{R}$ defined by

$$p_x(z) = p(x + z\,\zeta_0)$$

is monotone, where we put

$$\Omega_x := \{z \in \mathbb{R} \,:\, x + z\zeta_0 \in \Omega\}.$$

We give an example of a function which verifies such condition.

Example 3.3. Let

$$\mathbb{R}^N_+ := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^N : x_i > 0 \text{ for all } i = 1, \cdots, n\}$$

and Ω be a bounded domain of $\mathbb{R}^N$ contained in $\mathbb{R}^N_+$. Also, let $\zeta_0 = (0,1,0,\cdots,0)$. Then, the function $p \in C(\overline{\Omega})$ defined by

$$p((x_1, x_2, \cdots, x_n)) := 5 + x_2 \quad \text{for all } (x_1, x_2, \cdots, x_n) \in \overline{\Omega}$$

satisfies hypothesis (H3).

We underline that from Theorem 3.3 of [7], we know that hypothesis (H3) guarantees that

$$\inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\int_\Omega |u|^{p(x)}\,\mathrm{d}x} > 0.$$

According to this, we have that there exists $c_0 > 0$ such that

$$\int_\Omega |u|^{p(x)}\,\mathrm{d}x \le c_0 \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \tag{3.1}$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

4. **Existence result.** In this section, we present and prove our main result. Precisely, we show that, under very general assumptions on the exponents and the nonlinearity, problem (1.1) admits at least one sign-changing solution in $W_0^{1,\mathcal{T}}(\Omega)$. We underline that when we speak about a solution, we mean a weak solution. Thus, we say that $u \in W_0^{1,\mathcal{T}}(\Omega)$ is a solution of problem (1.1) if

$$\int_\Omega \mathcal{A}(u) \cdot \nabla w \,\mathrm{d}x + \int_\Omega |u|^{p(x)-2} uw \,\mathrm{d}x = \int_\Omega f(x,u)\, w \,\mathrm{d}x \tag{4.1}$$

is satisfied for all $w \in W_0^{1,\mathcal{T}}(\Omega)$. Now, let $\phi\colon W_0^{1,\mathcal{T}}(\Omega) \to \mathbb{R}$ be the functional defined by

$$\begin{aligned}\phi(u) = &\int_\Omega \left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu_1(x)}{q(x)}|\nabla u|^{q(x)} + \frac{\mu_2(x)}{r(x)}|\nabla u|^{r(x)}\right] \mathrm{d}x\\ &+ \int_\Omega \frac{1}{p(x)}|u|^{p(x)} \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x\end{aligned} \tag{4.2}$$

for all $u \in W_0^{1,\mathcal{T}}(\Omega)$, where $F(x,s) := \int_0^s f(x,t)\,\mathrm{d}t$. We note that ϕ is a C^1-functional with derivative given by

$$\langle \phi'(u), w\rangle = \int_\Omega \mathcal{A}(u) \cdot \nabla w \,\mathrm{d}x + \int_\Omega |u|^{p(x)-2} uw \,\mathrm{d}x - \int_\Omega f(x,u) w \,\mathrm{d}x \tag{4.3}$$

for all $u, w \in W_0^{1,\mathcal{T}}(\Omega)$. Consequently, according to (4.1), we have that the critical points of ϕ are solutions of problem (1.1).

In order to establish the existence of a sign-changing solution for problem (1.1), here we make use of variational tools and of so-called Nehari manifolds. We recall that the Nehari manifold corresponding to the functional ϕ is given by

$$\mathcal{N} := \left\{u \in W_0^{1,\mathcal{T}}(\Omega) \setminus \{0\} \text{ such that } \langle \phi'(u), u\rangle = 0\right\}.$$

From here, we see that if $u \neq 0$ is a critical point of ϕ, then $u \in \mathcal{N}$. As the critical points of ϕ are the solutions of problem (1.1) due to (4.1) and (4.3), we can affirm that $\mathcal{N}$ contains all the nontrivial solutions of problem (1.1). Now, we stress that $\mathcal{N}$ is much smaller than $W_0^{1,\mathcal{T}}(\Omega)$, and for this reason $\phi|_{\mathcal{N}}$ can have nice properties which fail to be true globally.

Next, we derive some such properties. Later, we will use them in order to produce our existence result.

Proposition 4.1. *Let hypotheses (H1) and (H2) be satisfied. Then, for any $u \in W_0^{1,\mathcal{T}}(\Omega) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$.*

Proof. Let $u \in W_0^{1,\mathcal{T}}(\Omega) \setminus \{0\}$ be fixed. In order to establish the claim, as a first step we show that there exists $t_u > 0$ such that $\langle \phi'(t_u u), t_u u\rangle = 0$. Clearly, according to the definition of Nehari manifold, this assures that $t_u u \in \mathcal{N}$.

Now, for $t > 0$ we have that

$$\begin{aligned}\langle \phi'(tu), u\rangle = &\int_\Omega [t^{p(x)-1}|\nabla u|^{p(x)} + \mu_1(x) t^{q(x)-1}|\nabla u|^{q(x)} + \mu_2(x) t^{r(x)-1}|\nabla u|^{r(x)}]\,\mathrm{d}x\\ &+ \int_\Omega t^{p(x)-1}|u|^{p(x)}\,\mathrm{d}x - \int_\Omega f(x,tu) u \,\mathrm{d}x.\end{aligned}$$

Further, on the base of hypothesis (H2)(iv), we can affirm that if $t \in (0,1)$, then

$$\langle \phi'(tu), u\rangle \geq \int_\Omega [t^{p(x)-1}|\nabla u|^{p(x)} + \mu_1(x) t^{q(x)-1}|\nabla u|^{q(x)} + \mu_2(x) t^{r(x)-1}|\nabla u|^{r(x)}]\,\mathrm{d}x$$

$$+\int_\Omega t^{p(x)-1}|u|^{p(x)}\,\mathrm{d}x - t^{r^+-1}\int_\Omega f(x,u)u\,\mathrm{d}x$$
$$\geq t^{p^+-1}\big[\rho_{p(\cdot)}(\nabla u)+\rho_{p(\cdot)}(u)\big] - t^{r^+-1}\int_\Omega f(x,u)u\,\mathrm{d}x.$$

Taking into account that $p^+ < r^+$ due to hypothesis (H1), from the previous inequality we deduce that

$$\langle \phi'(tu),u\rangle > 0 \quad \text{for } t\in(0,1) \text{ small enough.} \tag{4.4}$$

Next, we point out that for $t>1$, we have that

$$\frac{\langle \phi'(tu),u\rangle}{t^{r^+-1}} = \int_\Omega \left[\frac{1}{t^{r^+-p(x)}}|\nabla u|^{p(x)} + \frac{\mu_1(x)}{t^{r^+-q(x)}}|\nabla u|^{q(x)} + \frac{\mu_2(x)}{t^{r^+-r(x)}}|\nabla u|^{r(x)}\right]\mathrm{d}x$$
$$+\int_\Omega \frac{1}{t^{r^+-p(x)}}|u|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{f(x,tu)}{t^{r^+-1}|u|^{r^+-2}u}|u|^{r^+}\,\mathrm{d}x.$$

Also, according to hypothesis (H2)(ii), we know that

$$\int_\Omega \frac{f(x,tu)}{|tu|^{r^+-2}\,tu}|u|^{r^+}\,\mathrm{d}x \to +\infty \quad \text{as } t\to+\infty.$$

Keeping this in mind along with the fact that $p(x)<q(x)<r(x)\leq r^+$ for all $x\in\overline{\Omega}$ due to hypothesis (H1), if we pass to the limit as $t\to+\infty$ in the above equality, we derive that

$$\lim_{t\to+\infty}\frac{\langle \phi'(tu),u\rangle}{t^{r^+-1}} = -\infty,$$

that is, we have that

$$\langle \phi'(tu),u\rangle < 0 \quad \text{for } t>0 \text{ large enough.} \tag{4.5}$$

Thus, recalling that u is fixed and $\phi'(tu)$ is a continuous function of $t>0$, the intermediate value theorem, in accordance with (4.4) and (4.5), permits us to conclude that there exists $t_u>0$ such that

$$\langle \phi'(t_u u),u\rangle = 0.$$

From here, we easily derive that

$$\langle \phi'(t_u u),t_u u\rangle = 0$$

which means that $t_u u\in\mathcal{N}$.

Now, we must only show that such t_u is unique. With this purpose, we stress that if $t>0$ is such that $\langle \phi'(tu),u\rangle=0$, then we have that

$$\int_\Omega \left[\frac{1}{t^{r^+-p(x)}}|\nabla u|^{p(x)} + \frac{\mu_1(x)}{t^{r^+-q(x)}}|\nabla u|^{q(x)} + \frac{\mu_2(x)}{t^{r^+-r(x)}}|\nabla u|^{r(x)} + \right.$$
$$\left. + \frac{1}{t^{r^+-p(x)}}|u|^{p(x)} - \frac{f(x,tu)}{|tu|^{r^+-2}\,tu}|u|^{r^+}\right]\mathrm{d}x = 0.$$

As hypothesis (H2)(iv) holds, we know that the function

$$\frac{f(x,s)}{|s|^{r^+-1}}$$

is strictly increasing in $(-\infty,0)$ and $(0,+\infty)$ for a.a. $x\in\Omega$. Taking this into account along with the fact that $p(x)<q(x)<r(x)\leq r^+$ for all $x\in\overline{\Omega}$, we can affirm that the first member of the above equality is a strictly decreasing function of t. Therefore, there can be at most a single value of t for which the previous

equation holds. This means that there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$, and hence the claim is proved. $\square$

Proposition 4.2. *Let hypotheses (H1), (H2), and (H3) be satisfied. Then, for any $u \in \mathcal{N}$ and for all $t > 0$ with $t \neq 1$, we have $\phi(tu) < \phi(u)$.*

Proof. Let $u \in \mathcal{N}$ be fixed. We recall that according to hypotheses (H2)(i),(ii), we know that for any $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that

$$F(x,s) \geq \frac{\varepsilon}{r^+}|s|^{r^+} - c_\varepsilon$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$. Based on this, for $t > 1$, we can write that

$$\begin{aligned}\phi(tu) \leq & \frac{t^{q^+}}{p^-}\int_\Omega \left[|\nabla u|^{p(x)} + \mu_1(x)|\nabla u|^{q(x)}\right] \mathrm{d}x + \frac{t^{r^+}}{r^-}\int_\Omega \mu_2(x)|\nabla u|^{r(x)} \mathrm{d}x \\ & + \frac{t^{p^+}}{p^-}\int_\Omega |u|^{p(x)} \mathrm{d}x - \int_\Omega \left(\frac{\varepsilon}{r^+}t^{r^+}|u|^{r^+} - c_\epsilon\right) \mathrm{d}x \\ \leq & \frac{t^{q^+}}{p^-}\int_\Omega \left[|\nabla u|^{p(x)} + \mu_1(x)|\nabla u|^{q(x)} + |u|^{p(x)}\right] \mathrm{d}x \\ & + t^{r^+}\left[\frac{1}{r^-}\int_\Omega \mu_2(x)|\nabla u|^{r(x)} \mathrm{d}x - \frac{\varepsilon}{r^+}\int_\Omega |u|^{r^+} \mathrm{d}x\right] + c_\varepsilon\,|\Omega|.\end{aligned}$$

Now, we remark that as u is fixed, we also have that

$$\begin{aligned}&\int_\Omega \left[|\nabla u|^{p(x)} + \mu_1(x)|\nabla u|^{q(x)} + |u|^{p(x)}\right] \mathrm{d}x, \\ &\int_\Omega \mu_2(x)|\nabla u|^{r(x)} \mathrm{d}x \quad \text{and} \quad \int_\Omega |u|^{r^+} \mathrm{d}x\end{aligned}$$

are fixed. Therefore, as $\varepsilon > 0$ is arbitrary, we can choose ε big enough so that

$$\left[\frac{1}{r^-}\int_\Omega \mu_2(x)|\nabla u|^{r(x)} \mathrm{d}x - \frac{\varepsilon}{r^+}\int_\Omega |u|^{r^+} \mathrm{d}x\right] < 0.$$

According to such choice of ε, we get that

$$\phi(tu) \leq Ct^{q^+} - \tilde{C}t^{r^+} + c_\varepsilon\,|\Omega|$$

for some $C, \tilde{C} > 0$. Now, taking into account that hypothesis (H1) guarantees that $q^+ < r^+$, we can affirm that

$$\phi(tu) < 0 \quad \text{for } t > 0 \text{ large enough.} \tag{4.6}$$

Next, we point out that hypotheses (H2)(i), (iii) assure that for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that

$$F(x,s) \leq \frac{\varepsilon}{p(x)}|s|^{p(x)} + C_\varepsilon|s|^{l(x)}$$

for a.a. $x \in \Omega$ and all $s > 0$. Thus, using the previous inequality, we see that for $0 < t < 1$, it holds that

$$\begin{aligned}\phi(tu) \geq & \frac{t^{p^+}}{p^+}\int_\Omega |\nabla u|^{p(x)} \mathrm{d}x + \frac{t^{q^+}}{q^+}\int_\Omega \mu_1(x)|\nabla u|^{q(x)} \mathrm{d}x \\ & + \frac{t^{r^+}}{r^+}\int_\Omega \mu_2(x)|\nabla u|^{r(x)} \mathrm{d}x + \frac{t^{p^+}}{p^+}\int_\Omega |u|^{p(x)} \mathrm{d}x\end{aligned}$$

$$- \frac{t^{p^-}}{p^-}\varepsilon \int_\Omega |u|^{p(x)}\,\mathrm{d}x - C_\varepsilon \frac{t^{l^-}}{l^-}\int_\Omega |u|^{l(x)}\,\mathrm{d}x.$$

Now, we recall that, as $l(x) < p^*$ due to hypothesis (H2)(i), the embedding

$$W_0^{1,\mathcal{T}}(\Omega) \hookrightarrow L^{l(\cdot)}(\Omega)$$

is continuous, see Proposition 2.3. Therefore, let e_l be the best positive constant such that

$$\|u\|_l \leq e_l\|u\| \quad \text{for all } u \in W_0^{1,\mathcal{T}}(\Omega). \tag{4.7}$$

Using (3.1) and (4.7) along with Propositions 2.1, 2.2, and 2.3, we derive that

$$\begin{aligned}
\phi(tu) \geq& \left(\frac{t^{p^+}}{p^+} - \frac{t^{p^-}}{p^-}\varepsilon c_0\right)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{t^{q^+}}{q^+}\int_\Omega \mu_1(x)|\nabla u|^{q(x)}\,\mathrm{d}x\\
&+ \frac{t^{r^+}}{r^+}\int_\Omega \mu_2(x)|\nabla u|^{r(x)}\,\mathrm{d}x - C_\varepsilon\frac{t^{l^-}}{l^-}\int_\Omega |u|^{l(x)}\,\mathrm{d}x\\
\geq& \min\left\{\left(\frac{t^{p^+}}{p^+} - \frac{t^{p^-}}{p^-}\varepsilon\, c_0\right), \frac{t^{r^+}}{r^+}\right\}\rho_{\mathcal{T}}(\nabla u)\\
&- C_\varepsilon \frac{t^{l^-}}{l^-}\max\{(e_l\|u\|)^{l^-}, (e_l\|u\|)^{l^+}\}.
\end{aligned} \tag{4.8}$$

Thus, if we choose $\varepsilon > 0$ such that

$$\frac{t^{p^+}}{p^+} - \frac{t^{p^-}}{p^-}\varepsilon\, c_0 > \frac{t^{r^+}}{r^+}$$

that is, $\varepsilon < \frac{p^-(r^+t^{p^+} - p^+t^{r^+})}{t^{p^-}c_0p^+r^+}$, from (4.8) we get that

$$\begin{aligned}
\phi(tu) &\geq \frac{t^{r^+}}{r^+}\min\{\|u\|^{p^-}, \|u\|^{r^+}\} - C_\varepsilon\frac{t^{l^-}}{l^-}\max\{(e_l\|u\|)^{l^-}, (e_l\|u\|)^{l^+}\}\\
&\geq C\,t^{r^+} - \tilde{C}\,t^{l^-}
\end{aligned}$$

for some $C, \tilde{C} > 0$ (we stress that u is fixed, and then $\|u\|$ is fixed as well. Also, we recall that $C, \tilde{C}$ are arbitrary constants that may change from line to line). As $r^+ < l^-$ due to hypothesis (H2)(i), we have that

$$\phi(tu) > 0 \quad \text{for } t > 0 \text{ small enough.} \tag{4.9}$$

Taking into account that u is fixed and $\phi(tu)$ is a continuous function of $t \geq 0$, on the basis of inequalities (4.6) and (4.9), we can affirm that $\phi(tu)$ has a local maximizer. Now, we recall that $u \in \mathcal{N}$, and then $\langle \phi'(u), u\rangle = 0$. Also, according to Proposition 4.1, we know that $t_u = 1$ is the unique $t > 0$ such that $\langle \phi'(tu), tu\rangle = 0$. This in particular guarantees that $\phi(tu)$ as a function of t has a unique critical point, which is $t_u = 1$. Consequently, $t_u = 1$ is a global maximizer of $\phi(tu)$, which means that

$$\phi(tu) < \phi(u) \quad \text{for all } t > 0 \text{ with } t \neq 1$$

and thus the proof is complete. □

Proposition 4.3. *Let hypotheses (H1), (H2), and (H3) be satisfied. Then, $\phi\,|_{\mathcal{N}}$ is coercive.*

Proof. In order to establish the claim, we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}$ such that $\|u_n\| \to +\infty$ as $n \to +\infty$. Then, we set $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. As the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded, we know that it admits a subsequence, still denoted by y_n, such that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,\mathcal{T}}(\Omega) \quad \text{and } y_n \to y \quad \text{in } L^{p(\cdot)}(\Omega). \tag{4.10}$$

Now, our aim is to show that $y = 0$. To this end, we suppose by way of contradiction that $y \neq 0$. According to (3.1), we can write that

$$\begin{aligned}\phi(u_n) &\leq \frac{1}{p^-}\rho_{\mathcal{T}}(\nabla u_n) + \frac{1}{p^-}\int_\Omega |u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x,u_n)\,\mathrm{d}x\\ &\leq \frac{1+c_0}{p^-}\,\rho_{\mathcal{T}}(\nabla u_n) - \int_\Omega F(x,u_n)\,\mathrm{d}x.\end{aligned}$$

From here, using Proposition 2.2, we derive that for $\|u_n\| > 1$ it results that

$$\phi(u_n) \leq \frac{1+c_0}{p^-}\|u_n\|^{r^+} - \int_\Omega F(x,u_n)\,\mathrm{d}x.$$

Then, dividing by $\|u_n\|^{r^+}$, we in addition get that

$$\frac{\phi(u_n)}{\|u_n\|^{r^+}} \leq \frac{1+c_0}{p^-} - \int_\Omega \frac{F(x,u_n)}{|u_n|^{r^+}}|y_n|^{r^+}\,\mathrm{d}x.$$

This, in accordance with hypothesis (H2)(ii), as $\{y_n\}_{n\in\mathbb{N}}$ is bounded and $y \neq 0$, gives

$$\frac{\phi(u_n)}{\|u_n\|^{r^+}} \to -\infty \quad \text{as } n \to +\infty.$$

Taking into account that from the proof of Proposition 4.2 we know that $\phi(u_n) > 0$ for all $n \in \mathbb{N}$, we have a contradiction. Hence, it must be $y = 0$.

Now, we remark that according to Proposition 4.2, we have

$$\phi(u_n) > \phi(ty_n)$$

for all $t > 0$ with $t \neq \|u_n\|$. Keeping this in mind and using Proposition 2.2, we see that for $t > 1$, the following inequality holds

$$\begin{aligned}\phi(u_n) \geq& \frac{1}{r^+}\rho_{\mathcal{T}}(\nabla t y_n) + \frac{1}{p^+}\int_\Omega |ty_n|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x,ty_n)\,\mathrm{d}x\\ \geq& \frac{1}{r^+}\|ty_n\|^{p^-} - \int_\Omega F(x,ty_n)\,\mathrm{d}x\\ =& \frac{t^{p^-}}{r^+} - \int_\Omega F(x,ty_n)\,\mathrm{d}x\end{aligned}$$

(we stress that $\|y_n\| = 1$ and $t > 1$, thus we have that $\|ty_n\| > 1$). Now, from (4.10), according to the fact that $y = 0$, we have that

$$\int_\Omega F(x,ty_n)\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

For this reason, we can affirm that there exists $n_0 \in \mathbb{N}$ such that

$$\phi(u_n) \geq \frac{t^{p^-}}{2r^+} \quad \text{for all } n > n_0.$$

Taking into account that $t > 1$ is arbitrary and the subsequence principle yields the result for the whole sequence, this assures that

$$\phi(u_n) \to +\infty \quad \text{as} \quad n \to +\infty$$

and therefore the claim holds. ☐

We recall that we are interested in producing a sign-changing solution for problem (1.1). With this in mind, we introduce the subset of $W_0^{1,\mathcal{T}}(\Omega)$ defined as

$$\mathcal{N}_0 := \left\{u \in W_0^{1,\mathcal{T}}(\Omega) \text{ such that } u^+, -u^- \in \mathcal{N}\right\}.$$

Now, we can give the following additional properties of the functional ϕ.

Proposition 4.4. *Let hypotheses (H1), (H2), and (H3) be satisfied. Then, we have that*

$$\inf_{u\in\mathcal{N}} \phi(u) > 0 \quad \textit{and} \quad \inf_{u\in\mathcal{N}_0} \phi(u) > 0.$$

Proof. We point out that with similar arguments to those in the proof of Proposition 4.2, we can see that for any $u \in W_0^{1,\mathcal{T}}(\Omega)$, the inequality

$$\begin{aligned}\phi(u) \geq \min&\left\{\left(\frac{1}{p^+} - \frac{1}{p^-}\varepsilon\, c_0\right), \frac{1}{r^+}\right\} \min\{\|u\|^{p^-}, \|u\|^{r^+}\}\\ &- \frac{C}{l^-}\max\{(e_l\|u\|)^{l^-}, (e_l\|u\|)^{l^+}\}\end{aligned}$$

holds for ε small enough and for some $C > 0$. From here, we in particular deduce that for $u \in W_0^{1,\mathcal{T}}(\Omega)$ with $\|u\| < 1$, it results that

$$\phi(u) \geq C\|u\|^{r^+} - \tilde{C}\|u\|^{l^-}$$

for some $C, \tilde{C} > 0$ (we again recall that C and $\tilde{C}$ may change from line to line). Now, as $r^+ < l^-$ due to hypothesis (H2)(i), choosing $\vartheta \in (0, 1)$ small enough, we can further affirm that

$$\phi(u) > d > 0 \quad \text{for all } u \in\ W_0^{1,\mathcal{T}}(\Omega) \text{ with } \|u\| = \vartheta.$$

This clearly guarantees that

$$\inf_{u\,\in\, W_0^{1,\mathcal{T}}(\Omega),\, \|u\|=\vartheta} \phi(u) \geq d > 0.$$

Keeping this in mind, in accordance with Proposition 4.2, we derive that for all $u \in \mathcal{N}$, the inequality

$$\phi(u) \geq \phi\left(\frac{\vartheta}{\|u\|}u\right) \geq \inf_{u\,\in\, W_0^{1,\mathcal{T}}(\Omega),\, \|u\|=\vartheta} \phi(u) > 0$$

holds, and, consequently, we conclude that

$$\inf_{u\in\mathcal{N}} \phi(u) > 0.$$

Lastly, we recall that if $u \in \mathcal{N}_0$, then both u^+ and $-u^-$ belong to $\mathcal{N}$. Hence, for all $u \in \mathcal{N}_0$, we have that

$$\phi(u) = \phi(u^+) + \phi(-u^-) \geq 2 \inf_{u\in\mathcal{N}} \phi(u) > 0,$$

which implies

$$\inf_{u\in\mathcal{N}_0} \phi(u) > 0.$$

This proves the claim. ☐

Proposition 4.5. *Let hypotheses* (H1), (H2), *and* (H3) *be satisfied. Then, there exists* $u_0 \in \mathcal{N}_0$ *such that*

$$\phi(u_0) = \inf_{u \in \mathcal{N}_0} \phi(u).$$

Proof. In order to establish the claim, we start with a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ such that

$$\lim_{n\to+\infty} \phi(u_n) = \inf_{u\in\mathcal{N}_0} \phi(u).$$

As $u_n \in \mathcal{N}_0$ for all $n \in \mathbb{N}$, we know that both u_n^+ and $-u_n^-$ belong to $\mathcal{N}$ for all $n \in \mathbb{N}$. Thus, according to Proposition 4.4, we can affirm that

$$\phi(u_n^+) > 0 \quad \text{and} \quad \phi(-u_n^-) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Now, we recall that for all $n \in \mathbb{N}$ it results that

$$u_n = u_n^+ - u_n^-$$

and hence it follows that

$$\phi(u_n) = \phi(u_n^+) + \phi(-u_n^-)$$

for all $n \in \mathbb{N}$. All this, along with Proposition 4.3, permits us to conclude that both the sequences $\{u_n^+\}_{n\in\mathbb{N}}$ and $\{-u_n^-\}_{n\in\mathbb{N}}$ are bounded in $\mathcal{N}$. Consequently, we can suppose that

$$u_n^+ \rightharpoonup u^+ \text{ in } W_0^{1,\mathcal{T}}(\Omega) \quad \text{and} \quad -u_n^- \rightharpoonup u^- \text{ in } W_0^{1,\mathcal{T}}(\Omega)$$

and, as the embeddings $W_0^{1,\mathcal{T}}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and $W_0^{1,\mathcal{T}}(\Omega) \hookrightarrow L^{l(\cdot)}(\Omega)$ are compact, we have in addition that

$$u_n^+ \to u^+ \quad \text{in both } L^{p(\cdot)}(\Omega) \text{ and } L^{l(\cdot)}(\Omega) \tag{4.11}$$

and

$$-u_n^- \to u^- \quad \text{in both } L^{p(\cdot)}(\Omega) \text{ and } L^{l(\cdot)}(\Omega) \tag{4.12}$$

for some $u^+ \geq 0$ and $u^- \leq 0$. Now, our aim is to show that $u^+ \neq 0$ and $u^- \neq 0$. To this end, we point out that from $u_n^+ \in \mathcal{N}$ for all $n \in \mathbb{N}$, we deduce that the equality

$$0 = \langle \phi'(u_n^+), u_n^+ \rangle = \rho_{\mathcal{T}}(\nabla u_n^+) + \int_\Omega |u_n^+|^{p(x)} \,\mathrm{d}x - \int_\Omega f(x, u_n^+) u_n^+ \,\mathrm{d}x \tag{4.13}$$

is verified for all $n \in \mathbb{N}$. Also, using hypothesis (H2)(i) along with Hölder's inequality, we can derive that

$$\begin{aligned}
\left| \int_\Omega f(x, u_n^+)\, u_n^+ \, dx \right| &\leq \int_\Omega b_0 \,(1 + |u_n^+|^{l(x)-1})\, |u_n^+| \, dx \\
&\leq C \, \|u_n^+\|_{p(\cdot)} + \|(u_n^+)^{l(\cdot)-1}\|_{l'(\cdot)} \, \|u_n^+\|_{l(\cdot)} \\
&\quad \text{(for some } C > 0) \\
&\leq C \, \|u_n^+\|_{p(\cdot)} + [\rho_{l(\cdot)}(u_n^+)]^\gamma \, \|u_n^+\|_{l(\cdot)} \\
&\quad \text{(for some } \gamma > 0, \text{ according to Proposition (2.1))} \\
&\leq C \, \|u_n^+\|_{p(\cdot)} + \tilde{C} \, \|u_n^+\|_{l(\cdot)} \\
&\quad \text{(for some } C, \tilde{C} > 0).
\end{aligned}$$

This along with (4.11) permits us to affirm that if $u_n^+ \to 0$ in both $L^{p(\cdot)}(\Omega)$ and $L^{l(\cdot)}(\Omega)$ as $n \to +\infty$ (which means that $u^+ = 0$), then we have that

$$\left| \int_\Omega f(x, u_n^+)\, u_n^+ \, dx \right| \to 0 \quad \text{as } n \to +\infty. \tag{4.14}$$

Thus, according to (4.14) and (4.13), we see that if $u^+ = 0$, then

$$\rho_{\mathcal{T}}(\nabla u_n^+) \to 0 \quad \text{as } n \to +\infty.$$

Now, using Proposition 2.2 (jv), we have that

$$\|\nabla u_n^+\|_{\mathcal{T}} := \|u_n^+\| \to 0 \quad \text{as } n \to +\infty,$$

and hence we infer that

$$u_n^+ \to 0 \quad \text{in} \quad W_0^{1,\mathcal{T}}(\Omega) \quad \text{as } n \to +\infty.$$

This along with Proposition 4.4, as $u_n^+ \in \mathcal{N}$, gives that

$$0 < \inf_{u \in \mathcal{N}} \phi(u) \leq \phi(u_n^+) \to \phi(0) = 0 \quad \text{as } n \to +\infty,$$

which is a contradiction. For this reason, we conclude that it must be $u^+ \neq 0$. In an analogous way, using (4.12) instead of (4.11), we can also derive that $u^- \neq 0$.

Now, we stress that according to Proposition 4.1, there exist $t_{u^+}, t_{u^-} > 0$ such that

$$t_{u^+} u^+ \in \mathcal{N} \quad \text{and} \quad t_{u^-} u^- \in \mathcal{N}.$$

As both u^+ and u^- are different from zero, we know that

$$t_{u^+} u^+ > 0 \quad \text{and} \quad t_{u^-} u^- < 0.$$

Then, setting $u_0 = t_{u^+} u^+ + t_{u^-} u^-$, we have that $u_0^+ = t_{u^+} u^+$ and $-u_0^- = t_{u^-} u^-$, and hence it follows that $u_0 \in \mathcal{N}_0$. At this point, we note that the functional ϕ is sequentially weakly lower semicontinuous. With this in mind and using Proposition 4.2, we can write that

$$\begin{aligned}
\inf_{u \in \mathcal{N}_0} \phi(u) &= \lim_{n \to +\infty} \phi(u_n) \\
&= \lim_{n \to +\infty} [\phi(u_n^+) + \phi(-u_n^-)] \\
&\geq \liminf_{n \to +\infty} [\phi(t_{u^+} u_n^+) + \phi(-t_{u^-} u_n^-)] \\
&\geq \phi(t_{u^+} u^+) + \phi(t_{u^-} u^-) \\
&= \phi(u_0) \\
&\geq \inf_{u \in \mathcal{N}_0} \phi(u)
\end{aligned}$$

which gives

$$\phi(u_0) = \inf_{u \in \mathcal{N}_0} \phi(u).$$

So, the claim holds. □

Finally, we are in a position to state our main result. We stress that in order to do this, we follow the basic idea used in [11], and then in [3, 9].

Theorem 4.6. *Let hypotheses (H1), (H2), and (H3) be satisfied. Then, problem (1.1) has at least one sign-changing solution in* $W_0^{1,\mathcal{T}}(\Omega)$.

Proof. First, we point out that according to Proposition 4.5, there exists $u_0 \in \mathcal{N}_0$ such that

$$\phi(u_0) = \inf_{u \in \mathcal{N}_0} \phi(u).$$

Our goal is then to show that u_0 is a critical point of ϕ. In fact, from (4.1) and (4.3) and the definitions of $\mathcal{N}$ and $\mathcal{N}_0$, we derive that if $u_0 \in \mathcal{N}_0$ is a critical point of ϕ, then it is a nontrivial solution of problem (1.1) with both u_0^+ and u_0^- different from zero. Consequently, we have that u_0 is a sign-changing solution of problem (1.1), and therefore the claim holds.

In order to prove that $u_0 \in \mathcal{N}_0$ is a critical point of ϕ and then that $\phi'(u_0) = 0$, we make use of Lemma 2.4. First, we recall that as $u_0 \in \mathcal{N}_0$, we have that both u_0^+ and u_0^- belong to $\mathcal{N}$. Hence, according to Proposition 4.2, we know that the inequality

$$\begin{aligned} \phi(su_0^+ - tu_0^-) &= \phi(su_0^+) + \phi(-tu_0^-) \\ &< \phi(u_0^+) + \phi(-u_0^-) \\ &= \phi(u_0) \end{aligned} \tag{4.15}$$

holds for all $s, t > 0$ such that at least one between s and t is different from 1. At this point, we argue by way of contradiction and suppose that $\phi'(u_0) \neq 0$. This permits us to affirm that there exist $\delta, \nu > 0$ such that

$$\|\phi'(u)\|_* \geq \nu \text{ for all } u \in W_0^{1,\mathcal{T}}(\Omega) \text{ with } \|u - u_0\| \leq 3\delta. \tag{4.16}$$

Now, let δ be as given in (4.16) and $\xi \in (0,1)$ be such that

$$\|su_0^+ - tu_0^- - u_0\| < \delta \quad \text{for all } s, t \in (1-\xi, 1+\xi). \tag{4.17}$$

Thus, we set $D := (1-\xi, 1+\xi) \times (1-\xi, 1+\xi)$. According to (4.15), we have that

$$M := \max_{(s,t) \in \partial D} \phi(su_0^+ - tu_0^-) < \phi(u_0) = \inf_{u \in \mathcal{N}_0} \phi(u).$$

Then, setting

$$a_0 := \inf_{u \in \mathcal{N}_0} \phi(u) \quad \text{and} \quad \epsilon = \min\left\{\frac{a_0 - M}{4}, \frac{\nu\delta}{8}\right\},$$

from (4.16) we see that

$$\|\phi'(u)\|_* \geq \nu \geq \frac{8\epsilon}{\delta} \text{ for all } u \in W_0^{1,\mathcal{T}}(\Omega) \text{ with } \|u - u_0\| \leq 3\delta.$$

Hence, if we choose

$$S := \{u \in W_0^{1,\mathcal{T}}(\Omega) : \|u - u_0\| < \delta\},$$

we have that all the assumptions in Lemma 2.4 are satisfied. Consequently, according to Lemma 2.4, we can affirm there exists $\eta \in C([0,1] \times W_0^{1,\mathcal{T}}(\Omega), W_0^{1,\mathcal{T}}(\Omega))$ satisfying the following conditions:

$(j)'$ $\eta(1,u) = u$ if $u \notin \phi^{-1}([a_0 - 2\epsilon, a_0 + 2\epsilon]) \cap \{u \in W_0^{1,\mathcal{T}}(\Omega) : \|u - u_0\| < 3\,\delta\}$;

$(jj)'$ $\phi(\eta(1,u)) \leq a_0 - \epsilon$ for all $u \in W_0^{1,\mathcal{T}}(\Omega)$ with $\|u - u_0\| \leq \delta$ and $\phi(u) < a_0 + \epsilon$;

$(jjj)'$ $\|\eta(1,u) - u\| \leq \delta$ for all $u \in W_0^{1,\mathcal{T}}(\Omega)$.

In particular, from property $(jj)'$ of η, as (4.15) and (4.17) hold, we derive that

$$\max_{(s,t)\in D} \phi(\eta(1, su_0^+ - tu_0^-)) < a_0. \tag{4.18}$$

Also, we stress that, according to the definition of ϵ we have that

$$\phi(su_0^+ - tu_0^-) \leq M < a_0 - \frac{a_0 - M}{2} < a_0 - 2\epsilon \quad \text{for all } (s,t) \in \partial D.$$

This along with $(j)'$ permits us to affirm that

$$\eta(1, su_0^+ - tu_0^-) = su_0^+ - tu_0^- \quad \text{for all } (s,t) \in \partial D. \tag{4.19}$$

Now, we consider the mapping $h : \mathbb{R}_+ \times \mathbb{R}_+ \to W_0^{1,\mathcal{T}}(\Omega)$ defined by

$$h(s,t) = \eta(1, su_0^+ - tu_0^-)$$

and let $H_0, H_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}$ be the continuous functions given by

$$H_0(s,t) = (\langle \phi'(su_0^+), u_0^+ \rangle, \langle \phi'(-tu_0^-), -u_0^- \rangle)$$

and

$$H_1(s,t) = \left(\frac{1}{s} \langle \phi'(h^+(s,t)), h^+(s,t) \rangle, \frac{1}{t} \langle \phi'(-h^-(s,t)), -h^-(s,t) \rangle \right),$$

respectively. We stress that, according to (4.19), we have that

$$h(s,t) = su_0^+ - tu_0^- \quad \text{for all } (s,t) \in \partial D$$

which produces

$$H_0 = H_1 \quad \text{on } \partial D.$$

Also, from Proposition 4.1 and its proof, we see that

$$\langle \phi'(su_0^+), u_0^+ \rangle > 0 \quad \text{for all } 0 < s < 1,$$
$$\langle \phi'(su_0^+), u_0^+ \rangle < 0 \quad \text{for all } s > 1$$

and analogously

$$\langle \phi'(-tu_0^-), -u_0^- \rangle > 0 \quad \text{for all } 0 < t < 1,$$
$$\langle \phi'(-tu_0^-), -u_0^- \rangle < 0 \quad \text{for all } t > 1.$$

Hence, it follows that the Brouwer degree over D of the function H_0 at the value 0, denoted by $\deg(H_0, D, 0)$, is equal to 1. Now, as $H_0 = H_1$ on ∂D, from the dependence on the boundary values of the Brouwer degree (see [5]), we derive that

$$\deg(H_0, D, 0) = \deg(H_1, D, 0).$$

This, according to the existence property of the Brouwer degree (see [8, Theorem 4.11]), guarantees that

$$H_1(s,t) = (0,0) \text{ for some } (s,t) \in D.$$

Now, taking into account that $u_0^+ \neq 0$ and $u_0^- \neq 0$, from property $(jjj)'$ we derive that for $\delta > 0$ small enough it results that $h^+(s,t) \neq 0$ as well as $h^-(s,t) \neq 0$. Thus, with a view to the definition of H_1, we are able to affirm that

$$h(s,t) = \eta(1, su_0^+ - tu_0^-) \in \mathcal{N}_0 \quad \text{for some } (s,t) \in D.$$

Clearly, this is in contradiction with (4.18). Therefore, we conclude that u_0 is a critical point of ϕ, and hence the claim follows. □

Remark 4.7. Assume that hypotheses (H1), (H2), and (H3) are verified. Thus, let u_0 be a sign-changing solution for problem (1.1) as given in Theorem 4.6. We recall that this means that u_0 is a critical point of ϕ belonging to $\mathcal{N}_0$ such that $\phi(u_0) = \inf_{u\in\mathcal{N}_0}\phi(u)$. Also, we suppose that

$$f(x,s)s - r^+F(x,s) \geq 0$$

for all $s \in \mathbb{R}$ and for a.a. $x \in \Omega$. Then, we can affirm that u_0 has only two maximal regions (that is, there are only two disjoint open subsets of Ω) where it does not change sign. We can see this reasoning by way of contradiction. Thus, we suppose there are three disjoint open subsets of Ω, namely Ω_1, Ω_2, and Ω_3, on which u_0 has fixed sign. Now, for $i = 1, 2, 3$ we consider the functions

$$u_0^i(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega_i, \\ 0 & \text{if } x \in \Omega \setminus \Omega_i \end{cases}$$

and we suppose (it is not restrictive to do this) that

$$u_0^1|_{\Omega_1} > 0, \quad u_0^2|_{\Omega_2} < 0 \quad \text{and} \quad u_0^3|_{\Omega_3} > 0. \tag{4.20}$$

From (4.20), according to the fact that u_0 is a critical point of ϕ and $u_0 = (u_0^1 + u_0^2) + u_0^3$, taking into account that $(u_0^1+u_0^2)$ and u_0^3 have disjoint supports, we derive that

$$0 = \langle \phi'(u_0), u_0^3\rangle = \langle \phi'((u_0^1 + u_0^2) + u_0^3), u_0^3\rangle = \langle \phi'(u_0^3), u_0^3\rangle,$$

which means that

$$u_0^3 \in \mathcal{N}.$$

We point out that (4.20) in addition guarantees that $(u_0^1 + u_0^2)^+ = u_0^1$ and $(u_0^1 + u_0^2)^- = -u_0^2$. Therefore, again using (4.20) along with the fact that u_0 is a critical point of ϕ, we are able to see that

$$\begin{aligned}
0 &= \langle \phi'(u_0), (u_0^1 + u_0^2)^+\rangle \\
&= \langle \phi'((u_0^1 + u_0^2) + u_0^3), (u_0^1 + u_0^2)^+\rangle \\
&= \langle \phi'((u_0^1 + u_0^2)), (u_0^1 + u_0^2)^+\rangle + \langle \phi'(u_0^3, (u_0^1 + u_0^2)^+\rangle \\
&= \langle \phi'((u_0^1 + u_0^2)), (u_0^1 + u_0^2)^+\rangle \\
&= \langle \phi'((u_0^1 + u_0^2)^+), (u_0^1 + u_0^2)^+\rangle,
\end{aligned}$$

and similarly

$$\langle \phi'((u_0^1 + u_0^2)^-), (u_0^1 + u_0^2)^-\rangle = 0.$$

The previous equalities assure that both $(u_0^1 + u_0^2)^+$ and $-(u_0^1 + u_0^2)^-$ belong to $\mathcal{N}$, and consequently we have that

$$(u_0^1 + u_0^2) \in \mathcal{N}_0.$$

Now, with in mind that $(u_0^1 + u_0^2) \in \mathcal{N}_0$ and $u_0^3 \in \mathcal{N}$, in accordance with the fact that $f(x,s)s - r^+F(x,s) \geq 0$ for all $s \in \mathbb{R}$ and for a.a. $x \in \Omega$, we can write that

$$\begin{aligned}
\phi(u_0) &= \phi(u_0) - \frac{1}{r^+}\langle \phi'(u_0^3), u_0^3\rangle \\
&= \phi(u_0^1 + u_0^2) + \phi(u_0^3) - \frac{1}{r^+}\langle \phi'(u_0^3), u_0^3\rangle \\
&\geq \phi(u_0^1 + u_0^2) + \frac{1}{r^+}\rho_{\mathcal{T}}(\nabla u_0^3) + \frac{1}{p^+}\rho_{p(\cdot)}(u_0^3) - \int_\Omega F(x, u_0^3)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{r^+}\left[\rho_{\mathcal{T}}(\nabla u_0^3)+\rho_{p(\cdot)}(u_0^3)\right]+\frac{1}{r^+}\int_\Omega f(x,u_0^3)u_0^3\,\mathrm{d}x\\
&=\phi(u_0^1+u_0^2)+\left(\frac{1}{p^+}-\frac{1}{r^+}\right)\rho_{p(\cdot)}(u_0^3)\\
&\quad+\int_\Omega\left[\frac{1}{r^+}f(x,u_0^3)u_0^3-F(x,u_0^3)\right]\mathrm{d}x\\
&\geq\inf_{u\in\mathcal{N}_0}\phi(u)+\left(\frac{1}{p^+}-\frac{1}{r^+}\right)\rho_{p(\cdot)}(u_0^3).
\end{aligned}
$$

We note that, as $p^+ < r^+$ due to hypothesis (H1) and $u_0^3 \neq 0$, we have that

$$
\left(\frac{1}{p^+}-\frac{1}{r^+}\right)\rho_{p(\cdot)}(u_0^3)>0.
$$

This permits us to conclude that

$$
\phi(u_0)\geq\inf_{u\in\mathcal{N}_0}\phi(u)+\left(\frac{1}{p^+}-\frac{1}{r^+}\right)\rho_{p(\cdot)}(u_0^3)>\inf_{u\in\mathcal{N}_0}\phi(u)=\phi(u_0),
$$

that is, we arrive at a contradiction. Consequently, we can affirm that the disjoint open subsets of Ω on which u_0 has a fixed sign are only two, and therefore the claim holds.

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