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PARTIAL SUMS OF MEROMORPHIC FUNCTIONS LINKED WITH $q-$ DIFFERENTIAL OPERATOR

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ARTICLE INFO	ABSTRACT
Article history Received:2026-01-09 Received in revised form:2026-01-12 Accepted:2026-01-16 Available online *Keywords:* *Meromorphic functions;* *Partial sum,;* *q-calculus.* **2010 Mathematics Subject Classifications: 30C45**	*The paper presents the introduction of a novel linear differential operator for meromorphic functions associated with $q-$calculus. By means of this operator, a new subclass of meromorphic functions is defined and investigated in detail. The study focuses on deriving sufficient conditions that guarantee membership in this subclass and on establishing several analytic and geometric properties of the associated functions. Furthermore, the behavior of functions in the defined subclass is examined through the ratios of meromorphic functions to their sequences of partial sums. Various results describing these ratios are obtained, highlighting convergence and structural features. The findings enrich the theory of meromorphic functions and demonstrate the effectiveness of $q-$calculus-based operators in function theory.*

1. Introduction

Quantum calculus known as *q*-calculus is sometimes described as limitless calculus. It substitutes a difference operator for the classical derivative, allowing for the manipulation of sets of non-differentiable functions. Quantum difference operators play an intriguing role in a variety of mathematical fields, including the geometric function theory, calculus of variations, and relativity theory (see [1], [11], [19]). Gasper and Rahman's, and Kac and Cheung's books [9, 12] cover a large number of fundamental aspects of *q*-calculus.

We use the symbol Σ to represent the set of functions f that takes the following form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n \qquad (1)$$

that are analytic in the punctured open unit disk $U^*=\{z: z\in C: 0<|z|<1\}=U\setminus\{0\}$.

Tang et al. [20] introduced the *q*-derivative $D_q(f(z))$ for meromorphic functions, defined as follows:

$$D_q f(z)=\frac{f(z)-f(qz)}{(1-q)z}=-\frac{1}{qz^2}+\sum_{n=1}^{\infty}[n]_q a_n z^{n-1}, \quad (0<q<1). \quad (2)$$

We start with the definitions and various results from the $q-$analysis, including the $q-$factorial $[n]_q!$ for every non-negative integer $n \in \mathrm{N}$, which is characterized by

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [2]_q [1]_q; & (n \in \mathrm{N} \setminus \{1\}) \\ 1; & (n=1) \\ 0; & (n=0), \end{cases}$$

where

$$[n]_q = \frac{1-q^n}{1-q}. \qquad (3)$$

If $q \to 1^-$, then $[n]_q \to n$ and $\lim_{q\to 1^-} D_q f = f'$.

More recently, Ali et.al [2] introduced and studied the $q-$operator

$\mathrm{D}_{\rho,\chi}^{r,q} : \Sigma \to \Sigma \ \ (r \in \mathbb{N}_0, \rho, \chi \geq 0, 0 < q < 1)$ defined by

$\mathrm{D}_{\rho,\chi}^{0,q} f(z) := \mathrm{D}_{\rho,\chi}^{q} f(z) = f(z)$

$$\mathrm{D}_{\rho,\chi}^{1,q} f(z) = (1-\chi) f(z) + \frac{\chi}{[\rho]_q z^\rho} D_q(z^{\rho+1} f(z)) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{[\rho]_q + \chi\left([n+\rho+1]_q - [\rho]_q\right)}{[\rho]_q} a_n z^n$$

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$$\mathrm{D}_{\rho,\chi}^{r,q} f(z) = (1-\chi)\mathrm{D}_{\rho,\chi}^{r-1,q} f(z) + \frac{\chi}{[\rho]_q z^\rho} D_q\left(z^{\rho+1}\mathrm{D}_{\rho,\chi}^{r-1,q} f(z)\right).$$

Thus, we have the power series

$$\mathrm{D}_{\rho,\chi}^{r,q} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{[\rho]_q + \chi\left([n+\rho+1]_q - [\rho]_q\right)}{[\rho]_q} \right)^r a_n z^n. \qquad (4)$$

Building on the research conducted in [14,18], we introduce a subclass denoted as $MS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$, which is defined using the operator $\mathrm{D}_{\rho,\chi}^{r,q} f(z)$ in the following manner:

Definition 1.1. A function $f \in \Sigma$ is said to belong to the class $MS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$, if

$$\left| \frac{qzD_q\left(\mathrm{D}_{\rho,\chi}^{r,q} f(z)\right) + \mathrm{D}_{\rho,\chi}^{r,q} f(z)}{\mathrm{B}qzD_q\left(\mathrm{D}_{\rho,\chi}^{r,q} f(z)\right) + \mathrm{A}\mathrm{D}_{\rho,\chi}^{r,q} f(z)} \right| \leq 1 \quad \left(z \in \mathrm{U}^*\right), \ (5)$$

where $r \in \mathbb{N}_0, \rho, \chi \geq 0, 0 < q < 1$ and $-1 \leq \mathrm{B} < \mathrm{A} \leq 1$.

Furthermore, a function

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad \left(a_n > 0, z \in \mathrm{U}^*\right), \qquad (6)$$

belongs to the class $TMS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$, if it meets the requirement stated in equation (5).

It's worth noting that the previous definition is primarily inspired by the latest research by Morga [14] and Srivastava et al. [18]

This paper's primary aim is to give partial sums of functions that belong to the classes $MS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$ and $TMS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$. Unless specified otherwise, we'll assume that $r\in\mathbb{N}_0$, $\rho,\chi\ge 0$, $0<q<1$ and $-1\le \mathrm{A}<\mathrm{B}\le 1$ in this paper.

2. Coefficient Bonds

This section outlines the process of the sufficient conditions based on coefficient estimates for functions f that are part of the subclasses $MS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$ and $TMS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$.

Theorem 2.1. Let $f\in\Sigma$ as in (1) and satisfies the inequality

$$\sum_{n=1}^{\infty}\Big((\mathrm{B}+1)q[n]_q+\mathrm{A}+1\Big)\left(\frac{[\rho]_q+\chi\big([n+\rho+1]_q-[\rho]_q\big)}{[\rho]_q}\right)^r|a_n|<\mathrm{A}-\mathrm{B}, \quad (7)$$

then $f\in MS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$.

Proof. To prove that $f\in MS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$, we must show that

$$\left|qzD_q\left(\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right)+\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right|-\left|\mathrm{B}qzD_q\left(\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right)+\mathrm{A}\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right|\le 0. \quad (8)$$

By using the triangle inequality to the left of side of (8), we have

$$\begin{aligned}&\left|qzD_q\left(\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right)+\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right|-\left|\mathrm{B}qzD_q\left(\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right)+\mathrm{A}\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right|\\&=\left|\sum_{n=1}^{\infty}\left(q[n]_q+1\right)\left(\frac{[\rho]_q+\chi\big([n+\rho+1]_q-[\rho]_q\big)}{[\rho]_q}\right)^r a_nz^n\right|\\&-\left|(\mathrm{A}-\mathrm{B})\frac{1}{z}+\sum_{n=1}^{\infty}\left(\mathrm{A}+\mathrm{B}q[n]_q\right)\left(\frac{[\rho]_q+\chi\big([n+\rho+1]_q-[\rho]_q\big)}{[\rho]_q}\right)^r a_nz^n\right|\\&\le\sum_{n=1}^{\infty}\Big((\mathrm{B}+1)q[n]_q+\mathrm{A}+1\Big)\left(\frac{[\rho]_q+\chi\big([n+\rho+1]_q-[\rho]_q\big)}{[\rho]_q}\right)^r|a_n|-(\mathrm{A}-\mathrm{B}).\end{aligned} \quad (9)$$

Thus, from (7) and (9), we obtain that

$$\sum_{n=1}^{\infty}\Big((\mathrm{B}+1)q[n]_q+\mathrm{A}+1\Big)\left(\frac{[\rho]_q+\chi\big([n+\rho+1]_q-[\rho]_q\big)}{[\rho]_q}\right)^r|a_n|-(\mathrm{A}-\mathrm{B})\le 0,$$

which proves the theorem.

Theorem 2.2. Let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n\ (a_n\ge 0)$ in U^*. Then $f(z)\in TMS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$, if and only if inequality (7) is satisfied. The result is sharp for the function $f(z)$, which is defined as

$$f(z)=\frac{\mathrm{A}-\mathrm{B}}{\left((\mathrm{B}+1)q[n]_q+\mathrm{A}+1\right)\left(\frac{[\rho]_q+\chi\left([n+\rho+1]_q-[\rho]_q\right)}{[\rho]_q}\right)^r}z^n,\quad (n\geq 1). \qquad (10)$$

Proof. Considering Theorem 2.1, it's enough to prove the validity of the "if" component. Assume that $f(z)\in TMS_q[r,\rho,\chi;\mathrm{A},\mathrm{B}]$. Then, we have

$$\mathrm{Re}\left\{\frac{qzD_q\left(\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right)+\mathrm{D}^{r,q}_{\rho,\chi}f(z)}{\mathrm{B}qzD_q\left(\mathrm{D}^{r,q}_{\rho,\chi}f(z)\right)+\mathrm{A}\mathrm{D}^{r,q}_{\rho,\chi}f(z)}\right\}\geq -1\quad \left(z\in\mathrm{U}^*\right). \quad (11)$$

Since $\mathrm{Re}\, f(z)\leq |f(z)|$ for all $z\in\mathrm{U}^*$, then

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}\left(q[n]_q+1\right)\left(\frac{[\rho]_q+\chi\left([n+\rho+1]_q-[\rho]_q\right)}{[\rho]_q}\right)^r a_n z^{n+1}}{\mathrm{A}-\mathrm{B}+\sum_{n=1}^{\infty}\left(\mathrm{A}+\mathrm{B}q[n]_q\right)\left(\frac{[\rho]_q+\chi\left([n+\rho+1]_q-[\rho]_q\right)}{[\rho]_q}\right)^r a_n z^{n+1}}\right\}\leq 1' \qquad (12)$$

for all z and the above equation is true. By letting $z\to 1^-$ on the real axis, we have the following inequality

$$\sum_{n=1}^{\infty}\left(q[n]_q+1\right)\left(\frac{[\rho]_q+\chi\left([n+\rho+1]_q-[\rho]_q\right)}{[\rho]_q}\right)^r a_n$$
$$\leq \mathrm{A}-\mathrm{B}+\sum_{n=1}^{\infty}\left(\mathrm{A}+\mathrm{B}q[n]_q\right)\left(\frac{[\rho]_q+\chi\left([n+\rho+1]_q-[\rho]_q\right)}{[\rho]_q}\right)^r a_n .$$

Thus, we get the required inequality

$$\sum_{n=1}^{\infty}\left(q[n]_q(\mathrm{B}+1)+(\mathrm{A}+1)\right)\left(\frac{[\rho]_q+\chi\left([n+\rho+1]_q-[\rho]_q\right)}{[\rho]_q}\right)^r a_n\leq \mathrm{A}-\mathrm{B}.$$

This concludes the demonstration of our theorem.

3. Partial Sums

Inspired by previous studies that used the conventional idea of partial sums for analytic functions, such as Goodman [10], Silverman [16, 17], Murugusundaramoorthy and Velayudam [13], Darus and Ibrahim [5], Altıntas and Owa [3], Elhaddad et. al [8], and recently Deniz and co-auhors [4,6,7,15]. Our results related with partial sums as follows:

Theorem 3.1. Let $-1<\mathrm{A}\leq 0$. If $f\in\Sigma$ of the form (1) and

$s_k(z)=\frac{1}{z}+\sum_{n=1}^{k-1}a_n z^n \quad (k\geq 2)$. Suppose that

$$\sum_{n=1}^{\infty} c_n |a_n| \leq 1, \quad (13)$$

where

$$c_n = \frac{\big((1+\mathrm{A}) + q[n]_q (1+\mathrm{B})\big)}{(\mathrm{A}-\mathrm{B})} \left(\frac{[\rho]_q + \chi\big([n+\rho+1]_q - [\rho]_q\big)}{[\rho]_q} \right)^r .$$

Then, we have

1) $f(z) \in TMS_q[r, \rho, \chi; \mathrm{A}, \mathrm{B}]$.

2) $\operatorname{Re}\left\{ \frac{f(z)}{s_k(z)} \right\} > 1 - \frac{1}{c_{k-1}}$. (14)

3) $\operatorname{Re}\left\{ \frac{s_k(z)}{f(z)} \right\} > \frac{c_{k-1}}{1+c_{k-1}}$. (15)

The estimates are sharp.

Proof. 1): It is obvious that $\frac{1}{z} \in TMS_q[r, \rho, \chi; \mathrm{A}, \mathrm{B}]$. Thus from Theorem 2.1 and the condition (13), we have $N_{\mu,q}\left(\frac{1}{z}\right) \subseteq TMS_q[r, \rho, \chi; \mathrm{A}, \mathrm{B}]$. This gives $f(z) \in TMS_q[r, \rho, \chi; \mathrm{A}, \mathrm{B}]$.

2): It is easy to see that $1 < c_k < c_{k+1}$. Thus

$$\sum_{n=1}^{k-2} |a_n| + c_{k+1} \sum_{n=k-1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} c_n |a_n| \leq 1. \quad (16)$$

Let

$$h_1(z) = c_{k-1}\left\{ \frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{c_{k-1}}\right) \right\} = 1 + \frac{c_{k-1} \sum_{n=k-1}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-2} a_n z^{n+1}} .$$

It follows from (16) that

$$\left| \frac{h_1(z)-1}{h_1(z)+1} \right| \leq \frac{c_{k-1} \sum_{n=k-1}^{\infty} |a_n|}{2 - 2\sum_{n=1}^{k-2} |a_n| - c_{k-1} \sum_{n=k-1}^{\infty} |a_n|} \leq 1 \quad \left(z \in \mathrm{U}^*\right).$$

Therefore we obtain the inequality (14).

If we take

$$f(z) = \frac{1}{z} - \frac{z^{k-1}}{c_{k-1}}, \quad (17)$$

then

$$f(z)=1-\frac{z^k}{c_{k-1}}\to 1-\frac{1}{c_{k-1}} \text{ as } z\to 1^-.$$

This demonstrates that the bound in (14) is best possible for any *k*.

3): Similarly, assuming that we take

$$h_2(z)=(1+c_{k+1})\left\{\frac{s_k(z)}{f(z)}-\left(\frac{c_{k-1}}{1+c_{k-1}}\right)\right\}=1+\frac{(1+c_{k-1})\sum\limits_{n=k-1}^{\infty}a_n z^{n+1}}{1+\sum\limits_{n=0}^{\infty}a_n z^{n+1}}.$$

Then, we decude that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right|\le\frac{(1+c_{k-1})\sum\limits_{n=k-1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k-2}|a_n|+(1-c_{k-1})\sum\limits_{n=k-1}^{\infty}|a_n|}\le 1,\quad (z\in \mathrm{U}^*),$$

which yields (15). The estimate (15) is sharp with the extremal function $f(z)$ given by (17).

4. Conclusion

In the current study, a new subclass of meromophic functions associated with a $q-$differential operator has been introduced and investigated. Main results related with coefficient bounds and partial sums for functions belonging to this subclass

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